Looking at Separation Algebras with Boolean BI-eyes

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First presented at TCS 2014, Rome, Italy.

Separation Logic

- Introduced by Reynolds&O'Hearn 01 to model:
  - a **resource** logic
  - properties of the memory space (cells)
  - aggregation of cells into wider structures
- Combines:
  - classical logic connectives: $\wedge$, $\vee$, $\rightarrow$ ...
  - multiplicative conjunction, magic wand : $*$, $-\!\!*$
- Defined via Kripke semantics extended by:

$$m \Vdash A * B \quad \text{iff} \quad \exists a, b \text{ s.t. } a, b \triangleright m \text{ and } a \Vdash A \text{ and } b \Vdash B$$

Separation models, Separation Algebras

- Decomposition $a, b \triangleright m$ interpreted in various structures:
  - stacks in pointer logic (Reynolds&O'Hearn&Yang 01), $a \uplus b \subseteq m$
  - but also $a \uplus b = m$ (Calcagno&Yang&O'Hearn 01)
  - trees in spatial logics (Calcagno&Cardelli&Gordon 02) $a \mid b \equiv m$
- Additive $\rightarrow$ can be Boolean (pointwise) or intuitionistic
- Separation Algebra (SA) (Calcagno&O'Hearn&Yang 07) :
  - partial and cancellative commutative monoid
  - also, single units, indivisible units, disjointness

Boolean BI (BBI) and PASL

- BBI loosely defined by Pym as BI + $\{\neg\neg A \to A\}$
  - Kripke semantics by ND-monoids, Hilbert system (LW&G 06)
  - Display Logic based cut-free proof-system (Brotherston 09)
  - Structured Sequent proof-search (Park&Seo&Park 13)
  - Labelled sequents (Hóu&Tiu&Goré 13)
- Propositionnal Abstract Separation Logic (PASL)
  - based on separation algebras, partial monoids + ...
  - labelled tableaux (Larchey-W.&Galmiche 09, Larchey-W. 14)
  - labelled sequents (Hóu&Clouston&Goré&Tiu 14)
- Family of undecidable logics (LW&G 10, B&K 10)

Kripke semantics of **BBI&PASL** (i)

- Non-deterministic (or relational) monoid (ND) $(M, \circ, U)$
  - $\circ : M \times M \longrightarrow \mathcal{P}(M)$ and $U \subseteq M$
  - for $X, Y \in \mathcal{P}(M)$, $\boxed{X \circ Y = \{z \mid \exists x \in X, \exists y \in Y, z \in x \circ y\}}$
  - $x \circ U = \{x\}$ (neutrality)
  - $x \circ y = y \circ x$ (commutativity)
  - $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity)
  - $(\mathcal{P}(M), \circ, U)$ is a (usual) commutative monoid
- In some papers, $U = \{u\}$ is singleton (no impact on BBI)

Kripke semantics of BBI&PASL (ii)

- Boolean (pointwise) Kripke semantics extended by:

$$m \Vdash A * B \quad \text{iff} \quad \exists a, b \text{ s.t. } m \in a \circ b \text{ and } a \Vdash A \text{ and } b \Vdash B$$

$$m \Vdash A \mathrel{-\!\!*} B \quad \text{iff} \quad \forall a, b \ \ (b \in a \circ m \text{ and } a \Vdash A) \Rightarrow b \Vdash B$$

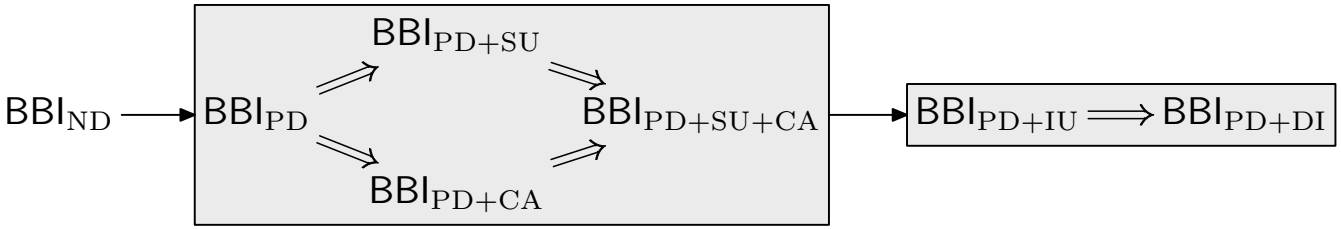
$$m \Vdash \mathbb{I} \quad \text{iff} \quad m \in U$$

- Validity in a ND-monoid $(M, \circ, U)$: $\forall \Vdash, \forall m,\ m \Vdash A$
- Validity in a sub-class $\mathcal{X} \subseteq \mathrm{ND}$: $\forall M \in \mathcal{X}, M \Vdash A$
- Set of formulas valid in $\mathcal{X}$: $\mathsf{BBI}_{\mathcal{X}}$
- $\mathcal{X} \subseteq \mathcal{Y}$ implies $\mathsf{BBI}_{\mathcal{Y}} \subseteq \mathsf{BBI}_{\mathcal{X}}$
- The full class ND: $\mathsf{BBI}_{\mathrm{ND}} \subseteq \mathsf{BBI}_{\mathcal{X}}$

Classes of models for **BBI**, SA properties

- Partial deterministic monoids (PD): $a \circ b \subseteq \{k\}$
- Total (deterministic) monoids (TD): $a \circ b = \{k\}$
- Single unit (SU): $\exists u \;\; U = \{u\}$
- Cancellative (CA): $\forall x, k, a, b \;\; x \in (k \circ a) \cap (k \circ b) \Rightarrow a = b$
- Indivisible units (IU): $\forall x, y \;\; x \circ y \cap U \neq \emptyset \Rightarrow x \in U$
- Disjointness (DI): $\forall x \;\; x \circ x \neq \emptyset \Rightarrow x \in U$
- Divisibility/splittability: $(x \notin U \Rightarrow \exists a, b \notin U, x \in a \circ b)$
- Cross-split property (longer definition ...)

Summary of our Results



- Some previous results:
  - $\mathsf{BBI}_{\mathrm{ND}} \subsetneq \mathsf{BBI}_{\mathrm{PD}}$ (Larchey-W.&Galmiche 10)
  - $\mathsf{BBI}_{\mathrm{PD}} \subsetneq \mathsf{BBI}_{\mathrm{PD+IU}}$ (Broth.&Villard 13, IU as a BBI formula)
  - most props cannot be axiomatized in BBI (Broth.&Villard 13)
- New results:
  - $\mathsf{BBI}_{\mathrm{PD}} = \mathsf{BBI}_{\mathrm{PD+SU+CA}}$
  - $\mathsf{BBI}_{\mathrm{PD+IU}} = \mathsf{BBI}_{\mathrm{PD+DI}}$

Single unit models/multiple unit model

- Consider any ND-monoid $(M, \circ, U)$
- Every element $x \in M$ has a unique unit $u_x \in U$ s.t. $x \circ u_x = \{x\}$
- If $x \in y \circ z$ then $u_x = u_y = u_z$
- The slice monoid:
  - $(M_x = \{y \in M \mid u_y = u_x\}, \circ \cap M_x \times M_x, \{u_x\})$ in class SU
  - $M = M_{u_1} \uplus \cdots \uplus M_{u_i} \uplus \cdots$
  - $M, x \nVdash F$ iff $M_x, x \nVdash F$ hence CM preserved by slicing
- For any class K closed under slicing: $\boxed{\mathsf{BBI}_K = \mathsf{BBI}_{K+\mathrm{SU}}}$
- In particular, $\mathsf{BBI}_{\mathrm{ND}} = \mathsf{BBI}_{\mathrm{SU}}$ and $\mathsf{BBI}_{\mathrm{PD}} = \mathsf{BBI}_{\mathrm{PD+SU}}$

Words and constraints based models for BBI

- **Resources as words** of $L^\star =$ multisets of letters, ϵ empty
- Constraints = (ordered) pairs of words: $m \rightarrowtail n$ with $m, n \in L^\star$
- Partial monoidal equivalence $\sim$ (PME) are closed under

$$\frac{}{\epsilon \rightarrowtail \epsilon}\ \langle\epsilon\rangle \qquad \frac{x \rightarrowtail y}{y \rightarrowtail x}\ \langle s\rangle \qquad \frac{x \rightarrowtail y \quad y \rightarrowtail z}{x \rightarrowtail z}\ \langle t\rangle$$

$$\frac{xy \rightarrowtail xy}{x \rightarrowtail x}\ \langle d\rangle \qquad \frac{ky \rightarrowtail ky \quad x \rightarrowtail y}{kx \rightarrowtail ky}\ \langle c\rangle$$

- Given $\mathcal{C}$, the closure is $\overline{\mathcal{C}} = \sim_{\mathcal{C}}$; compactness property
- PME extensions: $\sim + \{x_1 \rightarrowtail y_1, \ldots\} = \overline{\sim \cup \{x_1 \rightarrowtail y_1, \ldots\}}$
- Aka commutative Thue systems/symmetric VAS w/o reflexivity

PMEs represents the class PD + SU

- More pratical derived rules for PMEs:

$$\frac{kx \rightarrow y}{x \rightarrow x}\ \langle p_l \rangle \qquad \frac{x \rightarrow ky}{y \rightarrow y}\ \langle p_r \rangle \qquad \frac{x \rightarrow y \qquad yk \rightarrow m}{xk \rightarrow m}\ \langle e_l \rangle \qquad \frac{x \rightarrow y \qquad m \rightarrow yk}{m \rightarrow xk}\ \langle e_r \rangle$$

- From PME $\sim$ to PD + SU:
  - $\sim$ is a partial equivalence relation
  - $L^\star/\!\sim\ = \{[x] \mid x \sim x\}$ is a partial quotient
  - composition: $[z] \in [x] \bullet [y]$ iff $z \sim xy$; neutral: $[\epsilon]$
  - $(L^\star/\!\sim, \bullet, \{[\epsilon]\})$ of sub-class PD + SU
  - the map $\sim \mapsto L^\star/\!\sim$ is onto (surjective up to isomorphism)
- Any PD + SU monoid can be obtained as $L^\star/\!\sim$

Properties as extra PME rules

- Correspondence: CA/$\langle ca \rangle$, IU/$\langle iu \rangle$ and DI/$\langle di \rangle$

$$\frac{kx \dashv ky}{x \dashv y}\,\langle ca \rangle \qquad \frac{\epsilon \dashv xy}{\epsilon \dashv x}\,\langle iu \rangle \qquad \frac{xx \dashv xx}{\epsilon \dashv x}\,\langle di \rangle$$

- For any PME $\sim$:

$$\boxed{L^\star/\!\sim \text{ is of subclass CA iff } \sim \text{ closed under } \langle ca \rangle}$$

- Applies to IU/$\langle iu \rangle$ and DI/$\langle di \rangle$ as well
- but does not apply to any property (of course):
  - Divisibility/splittability
  - Cross-split property

Labelled tableaux for **BBI** and basic constraints

- Statements $(\mathbb{T}A : m)$, assertions $(\mathsf{ass} : \ m \twoheadrightarrow n)$

$\mathbb{T}\mathbb{I} : m$	$\mathbb{T}A * B : m$	$\mathbb{F}A \mathrel{-\!\!*} B : m$
$\vert$	$\vert$	$\vert$
$\mathsf{ass} : \ \epsilon \twoheadrightarrow m$	$\mathsf{ass} : \ \mathsf{ab} \twoheadrightarrow m$	$\mathsf{ass} : \ \mathsf{a}m \twoheadrightarrow \mathsf{b}$
	$\mathbb{T}A : \mathsf{a}$	$\mathbb{T}A : \mathsf{a}$
	$\mathbb{T}B : \mathsf{b}$	$\mathbb{F}B : \mathsf{b}$

- Basic extensions (Larchey-W.&Galmiche 09):
  1. $\sim + \{\epsilon \twoheadrightarrow m\}$ with $m \sim m$
  2. $\sim + \{\mathsf{ab} \twoheadrightarrow m\}$ with $m \sim m$ and $\mathsf{a} \neq \mathsf{b} \in L \backslash \mathcal{A}_\sim$
  3. $\sim + \{\mathsf{a}m \twoheadrightarrow \mathsf{b}\}$ with $m \sim m$ and $\mathsf{a} \neq \mathsf{b} \in L \backslash \mathcal{A}_\sim$

Constraints generated by proof-search

- The branches of (finite) tableaux contain basic extensions
- Simple PME (sPME) $= \infty$ sequence of basic extensions from $\emptyset$
- Exhaustive failed proof-search (saturated open branch):
  - contains a sPME, which provides a counter-model
  - strong completeness (Larchey-Wendling 14):

    $\mathsf{BBI}_{\mathsf{PD}} = \mathsf{BBI}_{\mathsf{PD+SU}}$ is complete for the class of simple PMEs
- For any sub-class K s.t. sPME $\subseteq K \subseteq$ PD: $\mathsf{BBI}_{\mathsf{PD}} = \mathsf{BBI}_{K}$
  - we study the properties of simple PMEs
  - obtain other refined completeness/equiv. results

Equations for (some) "free" PME extensions

- Given $\sim$ PME over L; m and α words in $L^\star$
- Hypotheses: $m \sim m$, $\mathcal{A}_\alpha \cap \mathcal{A}_\sim = \emptyset$ and $\boxed{\alpha \neq \epsilon}$
- We prove equations for type-1 and type-2 extensions:

$$\sim + \{\alpha \twoheadrightarrow m\} \;=\; \{x\delta\alpha^u \twoheadrightarrow y\delta\alpha^v \mid \exists i,\; xm^u \sim ym^v, xm^{i+u} \sim ym^{i+v} \text{ and } \delta \prec \alpha^i\}$$

$$\sim + \{\alpha m \twoheadrightarrow \alpha m\} \;=\; \sim \cup \{\delta x \twoheadrightarrow \delta y \mid x \sim y, \epsilon \neq \delta \prec \alpha \text{ and } \exists q\; xq \sim m\}$$

- For $\boxed{\alpha = \epsilon}$: first equation does not hold; second holds but useless
- Cover basic extensions $\mathsf{ab} \twoheadrightarrow m$, $\mathsf{a}m \twoheadrightarrow \mathsf{b}$
- But not basic ext. $\epsilon \twoheadrightarrow m$

Properties of $\epsilon \twoheadrightarrow m$, invertible letters and words

- PME closed under rules ($\epsilon \sim \gamma\beta$ means "γ and β are inverse"):

$$\frac{\epsilon \twoheadrightarrow \gamma \qquad \epsilon \twoheadrightarrow \beta}{\epsilon \twoheadrightarrow \gamma\beta}\ \langle \epsilon_c \rangle \qquad \frac{x \twoheadrightarrow y \qquad \epsilon \twoheadrightarrow \gamma\beta}{\gamma x \twoheadrightarrow \gamma y}\ \langle i_c \rangle \qquad \frac{x \twoheadrightarrow \beta y \qquad \epsilon \twoheadrightarrow \gamma\beta}{\gamma x \twoheadrightarrow y}\ \langle i_{\leftarrow} \rangle$$

$$\frac{\epsilon \twoheadrightarrow \gamma\beta \qquad \epsilon \twoheadrightarrow \gamma\beta'}{\beta \twoheadrightarrow \beta'}\ \langle i_{\uparrow} \rangle \qquad \frac{\gamma x \twoheadrightarrow \gamma y \qquad \epsilon \twoheadrightarrow \gamma\beta}{x \twoheadrightarrow y}\ \langle i_s \rangle \qquad \frac{\gamma x \twoheadrightarrow y \qquad \epsilon \twoheadrightarrow \gamma\beta}{x \twoheadrightarrow \beta y}\ \langle i_{\rightarrow} \rangle$$

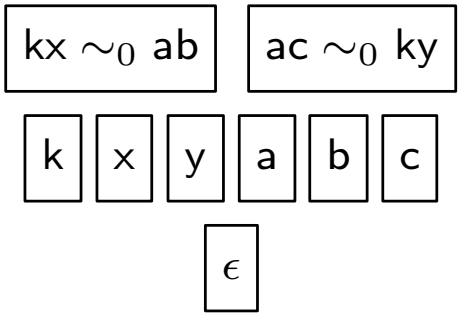
- Invertible letters: $\boxed{\mathcal{I}_\sim = \{\mathsf{c} \in L \mid \epsilon \sim \mathsf{c}\beta \text{ holds for some } \beta \in L^\star\}}$
  - invertible words: $\gamma \in \mathcal{I}_\sim^\star$ iff $\epsilon \sim \gamma\beta$ for some β
  - for any $\gamma \in \mathcal{I}_\sim^\star$, $x \sim y$ iff $\gamma x \sim \gamma y$
- If $\{x, y\} \cap \mathcal{I}_\sim^\star = \emptyset$ then $\mathcal{I}_{\sim + \{x \twoheadrightarrow y\}} = \mathcal{I}_\sim$
- If $\{x, y\} \cap \mathcal{I}_\sim^\star \neq \emptyset$ then $\mathcal{I}_\sim \cup \mathcal{A}_x \cup \mathcal{A}_y \subseteq \mathcal{I}_{\sim + \{x \twoheadrightarrow y\}}$

Cancellativity and invertible squares for PME ext.

- Cancellativity means closure under rule $\langle ca \rangle$: $\dfrac{kx \dashv ky}{x \dashv y}\,\langle ca \rangle$
- Invertible squares for $\sim$: for any c, if cc $\sim$ cc then c $\in \mathcal{I}_\sim$
- $\sim$ has $\langle ca \rangle$ then $\sim + \{\alpha \dashv m\}$ and $\sim + \{\alpha m \dashv \alpha m\}$ have $\langle ca \rangle$
- If α square-free (cc $\not\asymp \alpha$) then $\sim + \{\alpha \dashv m\}$ and $\sim + \{\alpha m \dashv \alpha m\}$ preserve invertible squares
- $\sim + \{\mathsf{ab} \dashv m\}$ is of the form $\sim + \{\alpha \dashv m\}$
- $\sim + \{\mathsf{a}m \dashv \mathsf{b}\} = (\sim + \{\mathsf{a}m \dashv \mathsf{a}m\}) + \{\mathsf{b} \dashv \mathsf{a}m\}$
  - thus $\sim + \{\mathsf{a}m \dashv \mathsf{b}\}$ of form $(\sim + \{\alpha m \dashv \alpha m\}) + \{\alpha' \dashv m\}$

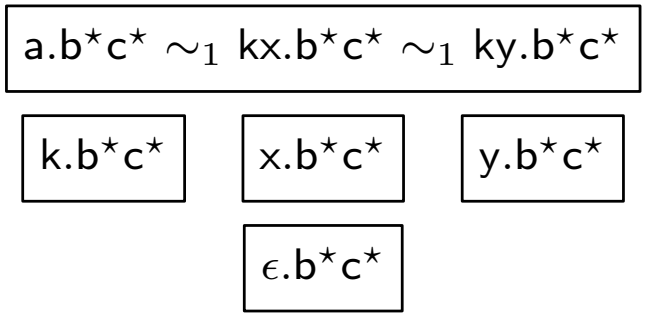
ab $\dashv m$ and a$m \dashv$ b ext. preserve cancel. and invert. squares

Extension $\sim + \{\epsilon \rightarrowtail m\}$



- $\mathcal{C}_0 = \{\mathsf{kx} \rightarrowtail \mathsf{ab}, \mathsf{ky} \rightarrowtail \mathsf{ac}\}$
- $\sim_0 = \overline{\mathcal{C}_0}$
- $\sim_0$ is cancellative, i.e. closed under $\langle ca \rangle$
- $\sim_0$ contains no squares, thus has invertible squares
- $\mathcal{I}_{\sim_0} = \emptyset$

Extension $\sim + \{\epsilon \rightarrowtail m\}$ does not preserve $\langle ca \rangle$



- $\mathcal{C}_1 = \mathcal{C}_0 \cup \{\epsilon \rightarrowtail \mathsf{b}, \epsilon \rightarrowtail \mathsf{c}\} = \{\mathsf{kx} \rightarrowtail \mathsf{ab}, \mathsf{ky} \rightarrowtail \mathsf{ac}\} \cup \{\epsilon \rightarrowtail \mathsf{b}, \epsilon \rightarrowtail \mathsf{c}\}$
- $\sim_1 = \overline{\mathcal{C}_1} = \sim_0 + \{\epsilon \rightarrowtail \mathsf{b}\} + \{\epsilon \rightarrowtail \mathsf{c}\}$
- $\mathcal{I}_{\sim_1} = \{\mathsf{b}, \mathsf{c}\}$
- $\sim_1$ is not cancellative, $\mathsf{kx} \sim_1 \mathsf{ky}$ but $\mathsf{x} \not\sim_1 \mathsf{y}$
- Cancellativity is **not preserved** by $\sim + \{\epsilon \rightarrowtail m\}$
- Squares are $\mathsf{bb} \sim_1 \mathsf{bb}$ and $\mathsf{cc} \sim_1 \mathsf{cc}$; thus $\sim_1$ has invertible squares

$\sim + \{\epsilon \rightarrowtail m\}$ does not preserve invertible squares

$$\boxed{\mathsf{a} \sim_2 \mathsf{k} \sim_2 \mathsf{ky} \sim_2 \cdots \sim_2 \mathsf{ky}^n \sim_2 \cdots}$$

$$\boxed{\epsilon} \quad \boxed{\mathsf{y}} \quad \boxed{\mathsf{y}^2} \cdots \boxed{\mathsf{y}^n} \cdots$$

- $\mathcal{C}_2 = \mathcal{C}_1 \cup \{\epsilon \rightarrowtail \mathsf{x}\} = \{\mathsf{kx} \rightarrowtail \mathsf{ab}, \mathsf{ky} \rightarrowtail \mathsf{ac}\} \cup \{\epsilon \rightarrowtail \mathsf{b}, \epsilon \rightarrowtail \mathsf{c}, \epsilon \rightarrowtail \mathsf{x}\}$
- $\sim_2 = \overline{\mathcal{C}_2} = \sim_1 + \{\epsilon \rightarrowtail \mathsf{x}\}$
- $\mathcal{I}_{\sim_2} = \{\mathsf{b}, \mathsf{c}, \mathsf{x}\}$; $\quad \epsilon \sim_2 \mathsf{b}^\star\mathsf{c}^\star\mathsf{x}^\star$ not displayed
- $\sim_2$ is not cancellative, $\mathsf{k} \sim_2 \mathsf{yk}$ but $\epsilon \nsim_2 \mathsf{y}$
- $\sim_2$ contains non-invertible squares, $\mathsf{yy} \sim_2 \mathsf{yy}$ but $\mathsf{y} \notin \mathcal{I}_{\sim_2}$
- Invertible squares prop. is $\boxed{\text{not preserved}}$ by $\sim + \{\epsilon \rightarrowtail m\}$

Group PME vs. Abelian groups

- In a group PME: $\boxed{\mathcal{I}_\sim = \mathcal{A}_\sim}$
- All defined letters are invertible
- Typical example: $\sim_{\mathcal{C}} = \overline{\mathcal{C}}$ with $\mathcal{C} = \{\epsilon \rightarrowtail a_1, \ldots, \epsilon \rightarrowtail a_p\}$
- If $\sim$ is a group PME then:
  - $\mathcal{L}_\sim = \mathcal{I}_\sim^\star$ ($x \sim x$ iff $x \in \mathcal{I}_\sim^\star$)
  - $L^\star/\sim$ is an Abelian group
  - $x \sim_{\mathcal{C}} y$ iff $x - y \in \sum_i \mathbb{Z} a_i$ ($\mathbb{Z}$-module)
- Group PMEs are cancellative
- Group PMEs have invertible squares (obvious)

Primary PME

- For $\sim$ PME, $m \sim m$, $\mathcal{A}_\sim \cap \mathcal{A}_\alpha = \emptyset$, $\alpha \neq \epsilon$ square-free
  - type-1 extension: $\sim + \{\alpha \nrightarrow m\}$
  - type-2 extension: $\sim + \{\alpha m \nrightarrow \mathsf{b}\}$ with $\mathsf{b} \in L \setminus (\mathcal{A}_\sim \cup \mathcal{A}_\alpha)$
- A primary PME is either (inductively)
  - a group-PME
  - a type-1 or type-2 extension of a primary PME
- Group-PME are cancellative and have invertible squares
- Primary extensions preserve both properties

Primary PMEs are cancellative and have invertible squares

From Basic PMEs to Primary PMEs (principle)

- Let $\mathcal{C} = x_1 \rightarrowtail y_1, \ldots, x_p \rightarrowtail y_p$ sequence of basic extensions
- Property: if $x \sim_{\mathcal{C}} y$ then $x \in \mathcal{I}_{\mathcal{C}}^{\star}$ iff $y \in \mathcal{I}_{\mathcal{C}}^{\star}$
- $\mathcal{I}_{\mathcal{C}}$ by a fixpoint computation
- Put invertible constraints $(x_i, y_i \in \mathcal{I}_{\mathcal{C}}^{\star})$ upfront:
  - $\mathcal{C} = \mathcal{D}, \mathcal{E}$ with $\mathcal{D} \in \mathcal{I}_{\mathcal{C}}^{\star} \times \mathcal{I}_{\mathcal{C}}^{\star}$ (hence $\sim_{\mathcal{D}}$ is a group-PME)
  - order in $\mathcal{E}$ same as in $\mathcal{C}$; $\mathcal{E} = \mathcal{C} \backslash \mathcal{D}$
  - $\mathcal{E}$ equivalent to a sequence of primary extensions of $\sim_{\mathcal{D}}$

From Basic PMEs to Primary PMEs (example)

- $\mathcal{C} = \mathsf{a}\epsilon \rightarrowtail \mathsf{b}, \mathsf{cd} \rightarrowtail \mathsf{b}, \mathsf{ec} \rightarrowtail \mathsf{f}, \epsilon \rightarrowtail \mathsf{f}$ (basic sequence)
- $\mathcal{I}_{\mathcal{C}} = \{\mathsf{f}, \mathsf{e}, \mathsf{c}\}$ and $\mathcal{D} = \epsilon \rightarrowtail \mathsf{f}, \mathsf{ec} \rightarrowtail \mathsf{f}$
- $\mathcal{E} = \mathsf{a} \rightarrowtail \mathsf{b}, \mathsf{cd} \rightarrowtail \mathsf{b}$
- But c (of $\mathsf{cd} \rightarrowtail \mathsf{b}$) is not new anymore w.r.t. $\epsilon \rightarrowtail \mathsf{f}, \mathsf{ec} \rightarrowtail \mathsf{f}, \mathsf{a} \rightarrowtail \mathsf{b}$
- $\epsilon \sim_{\mathcal{D}} \mathsf{ec}$ hence $\mathsf{cd} \rightarrowtail \mathsf{b}$ equiv. to $\mathsf{d} \rightarrowtail \mathsf{eb}$ (of type-1)
- $\mathcal{C}$ equiv. $\epsilon \rightarrowtail \mathsf{f}, \mathsf{ec} \rightarrowtail \mathsf{f}, \mathsf{a} \rightarrowtail \mathsf{b}, \mathsf{d} \rightarrowtail \mathsf{eb}$ which is primary

Simple PMEs and Equivalence Results

- Primary PMEs are cancellative with invertible squares
- Basic PMEs can be transformed into primary PMEs
  - hence basic PMEs are cancellative with invertible squares
  - simple PMEs are cancel. with invert. squares (by compactness)
- For any PME: $\langle iu \rangle$ and invertible squares implies $\langle di \rangle$
  - simple PMEs with $\langle iu \rangle$ also satisfy $\langle di \rangle$
- $\mathsf{BBI}_{\mathrm{PD}} = \mathsf{BBI}_{\mathrm{PD+SU}}$ is complete for simple PMEs:
  - hence $\mathsf{BBI}_{\mathrm{PD}} = \mathsf{BBI}_{\mathrm{PD+SU}} = \mathsf{BBI}_{\mathrm{PD+CA}} = \mathsf{BBI}_{\mathrm{PD+SU+CA}}$
  - and $\mathsf{BBI}_{\mathrm{PD+IU}} = \mathsf{BBI}_{\mathrm{PD+DI}} = \mathsf{BBI}_{\mathrm{PD+SU+CA+IU+DI}}$

Conclusion, Perspectives

- Labelled tableaux are sound/complete for PASL (PD + SU + CA)
- Cancellativity rule is redundant in labelled sequents for PASL
- $\mathsf{BBI}_{\mathrm{PD+SU+IU}}$ complete for disjointness DI
- Perspectives:
  - study other properties of sPMEs
  - provide a constructive proof of equivalence
  - effectively (efficiently ?) compute basic PMEs (proof assistant)