Sequent calculi for modal and epistemic logics

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Paolo Maffezioli and Sara Negri *A Gentzen-style analysis of Public Announcement Logic*. Proceedings of the International Workshop on Logic and Philosophy of Knowledge, Communication and Action, pp. 293-313, University of the Basque Country Press, 2010.

Public Announcement Logic

Morphology: $p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid \mathcal{K}_a B \mid [A]B$

Informal semantic:

$\mathcal{K}_a B$: "the agent a knows that B"

$[A]B$: "after the announcement of A, B holds"

Formal semantic: $M = (W, (R_a)_{a \in \mathcal{A}}, \Vdash)$

$$
\begin{array}{rcl}
w \Vdash p & & \text{given} \\
w \Vdash^{\varphi, A} p & := & w \Vdash^{\varphi} A \text{ and } w \Vdash^{\varphi} p \\
w \Vdash^{\varphi} \bot & & \text{never} \\
w \Vdash^{\varphi} A \vee B & := & w \Vdash^{\varphi} A \text{ or } w \Vdash^{\varphi} B \\
w \Vdash^{\varphi} A \wedge B & := & w \Vdash^{\varphi} A \text{ and } w \Vdash^{\varphi} B \\
w \Vdash^{\varphi} A \to B & := & w \Vdash^{\varphi} A \text{ implies } w \Vdash^{\varphi} B \\
w \Vdash^{\varphi} \mathcal{K}_a B & := & \text{for all } v, w\ R_a^{\varphi}\ v \text{ implies } v \Vdash^{\varphi} B \\
w \Vdash^{\varphi} [B]C & := & w \Vdash^{\varphi} B \text{ implies } w \Vdash^{\varphi, B} C
\end{array}
$$

Sequent calculi design

Principle

Main idea:	sequent proof system for “PAL semantic” theory
Right rules:	proof search in G3c
Left rules:	apply a “geometric implications” schema

Sequent calculi design

G3c: contraction and cut free sequent calculus for classical predicate logic

$$\frac{}{p, \Gamma \vdash \Delta, p}\ \text{ax} \qquad \frac{}{\bot, \Gamma \vdash \Delta}\ \text{L}\bot$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \text{L}\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \text{R}\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \text{L}\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \text{R}\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta}\ \text{L}\rightarrow \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}\ \text{R}\rightarrow$$

$$\frac{A[x \leftarrow t], \forall x A, \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\ \text{L}\forall \qquad \frac{\Gamma \vdash \Delta, A[x \leftarrow y]}{\Gamma \vdash \Delta, \forall x A}\ \text{R}\forall \quad y \notin \mathcal{FV}(\Gamma, \Delta)$$

$$\frac{A, [x \leftarrow y]\Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\ \text{L}\exists \quad y \notin \mathcal{FV}(\Gamma, \Delta) \qquad \frac{\Gamma \vdash \Delta, \exists x A, A[x \leftarrow t]}{\Gamma \vdash \Delta, \exists x A}\ \text{R}\exists$$

Sequent calculi design

Right rules example

Axiom: $w \Vdash^{\varphi} \mathcal{K}_a B := \forall v \quad w \; R_a^{\varphi} \; v \to v \Vdash^{\varphi} B$

Proof-search:

$$\cfrac{\cfrac{\cfrac{w \; R_a^{\varphi} \; v, \Gamma \vdash \Delta, v \Vdash^{\varphi} B}{\Gamma \vdash \Delta, w \; R_a^{\varphi} \; v \to v \Vdash^{\varphi} B} \text{R}\!\to}{\Gamma \vdash \Delta, \forall v \; w \; R_a^{\varphi} \; v \to v \Vdash^{\varphi} B} \text{R}\forall \quad v \notin \mathcal{FV}(\Gamma, \Delta)}{\Gamma \vdash \Delta, w \Vdash^{\varphi} \mathcal{K}_a B} \text{unfold}$$

Final rule:

$$\cfrac{w \; R_a^{\varphi} \; v, \Gamma \vdash \Delta, v \Vdash^{\varphi} B}{\Gamma \vdash w \Vdash^{\varphi} \Delta, \mathcal{K}_a B} \text{R}\mathcal{K} \quad v \notin \mathcal{FV}(\Gamma, \Delta)$$

Sequent calculi design

Left rules principle

$$\forall \vec{z}\,[P_1 \wedge \ldots \wedge P_m \rightarrow \exists \vec{x_1}(Q_{11} \wedge \ldots \wedge Q_{1r_1}) \vee \ldots \vee \exists \vec{x_n}(Q_{n1} \wedge \ldots \wedge Q_{nr_n})]$$

with P_i and Q_{jk} atoms, transformed to a left rule

$$\frac{\begin{array}{c} Q_{11}[\vec{x_1} \leftarrow \vec{y_1}\,], \ldots, Q_{1r_1}[\vec{x_1} \leftarrow \vec{y_1}\,], P_1, \ldots, P_m, \Gamma \vdash \Delta \\ \vdots \\ Q_{n1}[\vec{x_n} \leftarrow \vec{y_n}\,], \ldots, Q_{nr_n}[\vec{x_n} \leftarrow \vec{y_n}\,], P_1, \ldots, P_m, \Gamma \vdash \Delta \end{array}}{P_1, \ldots, P_m, \Gamma \vdash \Delta}$$

where $\vec{y_i} \notin \mathcal{FV}(P_1, \ldots, P_m, \Gamma, \Delta)$

Sequent calculi design

Left rule example

Principle

$$\forall \vec{z}(P_1 \wedge \ldots \wedge P_m \rightarrow \exists \vec{x_1}(Q_{11} \wedge \ldots \wedge Q_{1r_1}) \vee \ldots \vee \exists \vec{x_n}(Q_{n1} \wedge \ldots \wedge Q_{nr_n}))$$

$$\Downarrow$$

$$\frac{\begin{array}{c} Q_{11}[\vec{x_1} \leftarrow \vec{y_1}], \ldots, Q_{1r_1}[\vec{x_1} \leftarrow \vec{y_1}], P_1, \ldots, P_m, \Gamma \vdash \Delta \\ \vdots \\ Q_{n1}[\vec{x_n} \leftarrow \vec{y_n}], \ldots, Q_{nr_n}[\vec{x_n} \leftarrow \vec{y_n}], P_1, \ldots, P_m, \Gamma \vdash \Delta \end{array}}{P_1, \ldots, P_m, \Gamma \vdash \Delta}$$

Axiom $\forall w \quad w \Vdash^{\varphi} \mathcal{K}_a B \rightarrow \forall v \quad w \ R_a^{\varphi} \ v \rightarrow v \Vdash^{\varphi} B$ eq. to

$$\forall w \forall v \quad w \Vdash^{\varphi} \mathcal{K}_a B \wedge w \ R_a^{\varphi} \ v \rightarrow v \Vdash^{\varphi} B$$

giving

$$\frac{v \Vdash^{\varphi} B, w \Vdash^{\varphi} \mathcal{K}_a B, w \ R_a^{\varphi} \ v, \Gamma \vdash \Delta}{w \Vdash^{\varphi} \mathcal{K}_a B, w \ R_a^{\varphi} \ v, \Gamma \vdash \Delta} \mathsf{L}\mathcal{K}$$

Sequent calculus for PAL

$$\frac{}{w \Vdash p, \Gamma \vdash \Delta, w \Vdash p}\ \text{ax} \qquad \frac{}{w \Vdash^{\varphi} \bot, \Gamma \vdash \Delta}\ \mathsf{L}\bot^{\varphi}$$

$$\frac{w \Vdash^{\varphi} A, w \Vdash^{\varphi} P, \Gamma \vdash \Delta}{w \Vdash^{\varphi, \boldsymbol{A}} P, \Gamma \vdash \Delta}\ \mathsf{LAt}^{\varphi,\boldsymbol{A}} \qquad \frac{\Gamma \vdash \Delta, w \Vdash^{\varphi} A \quad \Gamma \vdash \Delta, w \Vdash^{\varphi} P}{\Gamma \vdash \Delta, w \Vdash^{\varphi, \boldsymbol{A}} P}\ \mathsf{RAt}^{\varphi,\boldsymbol{A}}$$

$$\frac{w \Vdash^{\varphi} B, w \Vdash^{\varphi} C, \Gamma \vdash \Delta}{w \Vdash^{\varphi} B \wedge C, \Gamma \vdash \Delta}\ \mathsf{L}\wedge^{\varphi} \qquad \frac{\Gamma \vdash \Delta, w \Vdash^{\varphi} B \quad \Gamma \vdash \Delta, w \Vdash^{\varphi} C}{\Gamma \vdash \Delta, w \Vdash^{\varphi} B \wedge C}\ \mathsf{R}\wedge^{\varphi}$$

$$\frac{w \Vdash^{\varphi} B, \Gamma \vdash \Delta \quad w \Vdash^{\varphi} C, \Gamma \vdash \Delta}{w \Vdash^{\varphi} B \vee C, \Gamma \vdash \Delta}\ \mathsf{L}\vee^{\varphi} \qquad \frac{\Gamma \vdash \Delta, w \Vdash^{\varphi} B, w \Vdash^{\varphi} C}{\Gamma \vdash \Delta, w \Vdash^{\varphi} B \vee C}\ \mathsf{R}\vee^{\varphi}$$

$$\frac{\Gamma \vdash \Delta, w \Vdash^{\varphi} B \quad w \Vdash^{\varphi} C, \Gamma \vdash \Delta}{w \Vdash^{\varphi} B \to C, \Gamma \vdash \Delta}\ \mathsf{L}{\to}^{\varphi} \qquad \frac{w \Vdash^{\varphi} B, \Gamma \vdash \Delta, w \Vdash^{\varphi} C}{\Gamma \vdash \Delta, w \Vdash^{\varphi} B \to C}\ \mathsf{R}{\to}^{\varphi}$$

$$\frac{v \Vdash^{\varphi} B, w \Vdash^{\varphi} \mathcal{K}_{\boldsymbol{a}} B, wR^{\varphi}_{\boldsymbol{a}} v, \Gamma \vdash \Delta}{w \Vdash^{\varphi} \mathcal{K}_{\boldsymbol{a}} B, wR^{\varphi}_{\boldsymbol{a}} v, \Gamma \vdash \Delta}\ \mathsf{L}\mathcal{K}^{\varphi}_{\boldsymbol{a}} \qquad \frac{wR^{\varphi}_{\boldsymbol{a}} v, \Gamma \vdash \Delta, v \Vdash^{\varphi} B}{\Gamma \vdash \Delta, w \Vdash^{\varphi} \mathcal{K}_{\boldsymbol{a}} B}\ \mathsf{R}\mathcal{K}^{\varphi}_{\boldsymbol{a}}$$

$$\frac{w \Vdash^{\varphi, \boldsymbol{B}} C, w \Vdash^{\varphi} [B]C, w \Vdash^{\varphi} B, \Gamma \vdash \Delta}{w \Vdash^{\varphi} [B]C, w \Vdash^{\varphi} B, \Gamma \vdash \Delta}\ \mathsf{L}[]^{\varphi} \qquad \frac{w \Vdash^{\varphi} B, \Gamma \vdash \Delta, w \Vdash^{\varphi, \boldsymbol{B}} C}{\Gamma \vdash \Delta, w \Vdash^{\varphi} [B]C}\ \mathsf{R}[]^{\varphi}$$

Labeled sequent calculus for PAL (G3PAL)

$$\dfrac{}{w : p, \Gamma \vdash \Delta, w : p}\ \mathsf{ax} \qquad \dfrac{}{w :^{\varphi} \bot, \Gamma \vdash \Delta}\ \mathsf{L}\bot^{\varphi}$$

$$\dfrac{w :^{\varphi} A, w :^{\varphi} P, \Gamma \vdash \Delta}{w :^{\varphi, A} P, \Gamma \vdash \Delta}\ \mathsf{LAt}^{\varphi, A} \qquad \dfrac{\Gamma \vdash \Delta, w :^{\varphi} A \quad \Gamma \vdash \Delta, w :^{\varphi} P}{\Gamma \vdash \Delta, w :^{\varphi, A} P}\ \mathsf{RAt}^{\varphi, A}$$

$$\dfrac{w :^{\varphi} B, w :^{\varphi} C, \Gamma \vdash \Delta}{w :^{\varphi} B \wedge C, \Gamma \vdash \Delta}\ \mathsf{L}\wedge^{\varphi} \qquad \dfrac{\Gamma \vdash \Delta, w :^{\varphi} B \quad \Gamma \vdash \Delta, w :^{\varphi} C}{\Gamma \vdash \Delta, w :^{\varphi} B \wedge C}\ \mathsf{R}\wedge^{\varphi}$$

$$\dfrac{w :^{\varphi} B, \Gamma \vdash \Delta \quad w :^{\varphi} C, \Gamma \vdash \Delta}{w :^{\varphi} B \vee C, \Gamma \vdash \Delta}\ \mathsf{L}\vee^{\varphi} \qquad \dfrac{\Gamma \vdash \Delta, w :^{\varphi} B, w :^{\varphi} C}{\Gamma \vdash \Delta, w :^{\varphi} B \vee C}\ \mathsf{R}\vee^{\varphi}$$

$$\dfrac{\Gamma \vdash \Delta, w :^{\varphi} B \quad w :^{\varphi} C, \Gamma \vdash \Delta}{w :^{\varphi} B \to C, \Gamma \vdash \Delta}\ \mathsf{L}\!\to^{\varphi} \qquad \dfrac{w :^{\varphi} B, \Gamma \vdash \Delta, w :^{\varphi} C}{\Gamma \vdash \Delta, w :^{\varphi} B \to C}\ \mathsf{R}\!\to^{\varphi}$$

$$\dfrac{v :^{\varphi} B, w :^{\varphi} \mathcal{K}_a B, wR_a^{\varphi} v, \Gamma \vdash \Delta}{w :^{\varphi} \mathcal{K}_a B, wR_a^{\varphi} v, \Gamma \vdash \Delta}\ \mathsf{L}\mathcal{K}_a^{\varphi} \qquad \dfrac{wR_a^{\varphi} v, \Gamma \vdash \Delta, v :^{\varphi} B}{\Gamma \vdash \Delta, w :^{\varphi} \mathcal{K}_a B}\ \mathsf{R}\mathcal{K}_a^{\varphi}$$

$$\dfrac{w :^{\varphi, B} C, w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma \vdash \Delta}{w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma \vdash \Delta}\ \mathsf{L}[]^{\varphi} \qquad \dfrac{w :^{\varphi} B, \Gamma \vdash \Delta, w :^{\varphi, B} C}{\Gamma \vdash \Delta, w :^{\varphi} [B]C}\ \mathsf{R}[]^{\varphi}$$

Labeled sequent calculus for PAL (G3PAL)

Results

Need for completion

$[A][B]C \leftrightarrow [A \wedge [A]B]C$ not derivable $\Longrightarrow$ 2 more rules:

$$\frac{w :^{\varphi,A,B} C, \Gamma \vdash \Delta}{w :^{\varphi,A\wedge[A]B} C, \Gamma \vdash \Delta}\, L_{cmp} \qquad \frac{\Gamma \vdash \Delta, w :^{\varphi,A,B} C}{\Gamma \vdash \Delta, w :^{\varphi,A\wedge[A]B} C}\, R_{cmp}$$

Structural properties

Weakening and contraction hp-admissible, cut rule admissible, all rules hp-invertible.

Automation

G3PAL sound and complete, *but* completeness proof relying on excluded middle and Koenig's lemma: G3PAL-based decision procedure ?

Comparison with tableaux method ?