Propositional dynamic logic with storing, recovering and parallel composition: decidability/undecidability results

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Contents

Syntax

Programs

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha ; \beta) \mid (\alpha \cup \beta) \mid \alpha^{\star} \mid (\alpha \parallel \beta)$

Formulas

- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

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Semantics

Models

- a model is a structure of the form $\mathcal{M} = (W, R, *, V)$ where
  - W is a nonempty set of states
  - R is a function $a \mapsto R(a) \subseteq W \times W$
  - $*$ is a ternary relation over W
  - V is a function $p \mapsto V(p) \subseteq W$

Truth conditions

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(p)^{\mathcal{M}} = V(p)$
  - $(\bot)^{\mathcal{M}}$ is empty
  - $(\neg\phi)^{\mathcal{M}} = W \setminus (\phi)^{\mathcal{M}}$
  - $(\phi \vee \psi)^{\mathcal{M}} = (\phi)^{\mathcal{M}} \cup (\psi)^{\mathcal{M}}$
  - $([\alpha]\phi)^{\mathcal{M}} = \{x\text{: for all } y \in W\text{, if } x(\alpha)^{\mathcal{M}}y\text{, } y \in (\alpha)^{\mathcal{M}}\}$

Semantics

Truth conditions according to Benevides *et al.* (2011)

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(a)^{\mathcal{M}} = R(a)$
  - $(\phi?)^{\mathcal{M}} = \{(x, y)\colon x = y \text{ and } y \in (\phi)^{\mathcal{M}}\}$
  - $(s_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y * (x, z)\}$
  - $(s_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y * (z, x)\}$
  - $(r_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x * (y, z)\}$
  - $(r_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x * (z, y)\}$
  - $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x(\alpha)^{\mathcal{M}} z \text{ and } z(\beta)^{\mathcal{M}} y\}$
  - $(\alpha \cup \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \cup (\beta)^{\mathcal{M}}$
  - $(\alpha^{\star})^{\mathcal{M}} = \{(x, y)\colon \text{there exists } n \in \mathbb{N} \text{ and there exists } z_0, \ldots, z_n \in W \text{ such that } x = z_0(\alpha)^{\mathcal{M}} \ldots (\alpha)^{\mathcal{M}} z_n = y\}$
  - $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z, t, u, v \in W \text{ such that } x * (z, t),\ y * (u, v),\ z(\alpha)^{\mathcal{M}} u \text{ and } t(\beta)^{\mathcal{M}} v\}$

Semantics

Truth conditions according to Frias (2002)

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(a)^{\mathcal{M}} = R(a)$
  - $(\phi?)^{\mathcal{M}} = \{(x, y)$: $x = y$ and $y \in (\phi)^{\mathcal{M}}\}$
  - $(s_1)^{\mathcal{M}} = \{(x, y)$: there exists $z \in W$ such that $y * (x, z)\}$
  - $(s_2)^{\mathcal{M}} = \{(x, y)$: there exists $z \in W$ such that $y * (z, x)\}$
  - $(r_1)^{\mathcal{M}} = \{(x, y)$: there exists $z \in W$ such that $x * (y, z)\}$
  - $(r_2)^{\mathcal{M}} = \{(x, y)$: there exists $z \in W$ such that $x * (z, y)\}$
  - $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)$: there exists $z \in W$ such that $x(\alpha)^{\mathcal{M}}z$ and $z(\beta)^{\mathcal{M}}y\}$
  - $(\alpha \cup \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \cup (\beta)^{\mathcal{M}}$
  - $(\alpha^{\star})^{\mathcal{M}} = \{(x, y)$: there exists $n \in \mathbb{N}$ and there exists $z_0, \ldots, z_n \in W$ such that $x = z_0(\alpha)^{\mathcal{M}} \ldots (\alpha)^{\mathcal{M}}z_n = y\}$
  - $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)$: there exists $z, t \in W$ such that $y * (z, t)$, $x(\alpha)^{\mathcal{M}}z$ and $x(\beta)^{\mathcal{M}}t\}$

Semantics

A model $\mathcal{M} = (W, R, *, V)$ is said to be separated iff

- if $x * (y, z)$ and $x * (t, u)$, $y = t$ and $z = u$

A model $\mathcal{M} = (W, R, *, V)$ is said to be distributive iff

- $u * (x, y)$ and $u * (z, t)$ iff $u * (x, t)$ and $u * (z, y)$

A model $\mathcal{M} = (W, R, *, V)$ is said to be $*$-deterministic iff

- if $x * (z, t)$ and $y * (z, t)$, $x = y$

A model $\mathcal{M} = (W, R, *, V)$ is said to be R-deterministic iff

- if $xR(a)y$ and $xR(a)z$, $y = z$

A model $\mathcal{M} = (W, R, *, V)$ is said to be serial iff

- $*(x, y)$ is nonempty

Expressivity

Programs

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha;\beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$

Formulas

- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

For all $i \in \{1, 2\}$ and for all s_i-free programs α

- the programs s_i and α are not equally interpreted in all separated models

For all $i \in \{1, 2\}$ and for all r_i-free programs α

- the programs r_i and α are not equally interpreted in all separated models

Expressivity

Programs

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha ; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$

Formulas

- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

For all atomic programs a, b and for all $\parallel$-free programs α

- the programs $a \parallel b$ and α are not equally interpreted in all separated models

A decision problem: satisfiability

Fragment of the language

- $\mathcal{L}$

Class of models

- $\mathcal{C}$

Decision problem

- $SAT(\mathcal{L}, \mathcal{C})$
  - input: a formula ϕ in $\mathcal{L}$
  - output: determine whether there exists a model $\mathcal{M}$ in $\mathcal{C}$ such that $(\phi)^{\mathcal{M}} \neq \emptyset$

Computability of $SAT(\mathcal{L}, \mathcal{C})$?

Decidability results

Within the class of all separated models

- *SAT* is in 2*EXPTIME* for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid (\alpha ; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$
- *SAT* is in *EXPTIME* for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid (\alpha ; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$
- *SAT* is in *PSPACE* for the fragment
  - $\alpha, \beta ::= a \mid s_1 \mid s_2 \mid (\alpha ; \beta) \mid (\alpha \cup \beta) \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Decidability results

Within the class of all separated $*$-deterministic models

- *SAT* is in *EXPTIME* for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Within the class of all distributive models

- *SAT* is in 2*EXPTIME* for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Undecidability results

Within the class of all separated models

- *SAT* is Σ^1_1-hard for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid r_1 \mid r_2 \mid (\alpha;\beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Within the class of all separated $*$-deterministic models

- *SAT* is Σ^1_1-hard for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid r_1 \mid r_2 \mid (\alpha;\beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Undecidability results

Within the class of all separated R-deterministic models

- SAT is Σ_1^1-hard for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Within the class of all separated R-deterministic $*$-deterministic models

- SAT is Σ_1^1-hard for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Undecidability results

Within the class of all separated serial models

- *SAT* is Σ_1^1-hard for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Within the class of all separated serial $*$-deterministic models

- *SAT* is Σ_1^1-hard for the fragment
  - $\alpha, \beta ::= a \mid \phi? \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
  - $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

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