Axiomatization/completeness of propositional dynamic logic with separation and parallel composition

Contents

Algebra of subrelations of an equivalence relation E on some set A

- set of subrelations of E that is closed for the following operations
  - 0, empty binary relation
  - $-R$, complement of a binary relation R wrt E
  - $R \cup S$, union of binary relations R and S
  - Id, identity binary relation on A
  - R^{-1}, transposition of a binary relation R
  - $R \circ S$, composition of binary relations R and S

Tarski (1954)

- the class of all algebra of binary relations is axiomatizable by a set of equations
- is the class of all algebra of binary relations axiomatizable by a finite set of equations ?

Monk (1964)

- NO

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In order to overcome this drawback

- an extra binary operation on relations called fork is added

Addition of fork has the following consequence

- the class of algebras obtained can be axiomatized by a finite set of equations

Introduction and motivations

Fork

Algebra of subrelations of an equivalence relation E on some set A closed under a binary function $*$

- $R \underline{\nabla} S$, fork of binary relations R and S

The definition of the operation fork is given by

- $R \underline{\nabla} S = \{(u, v * w) : \; uRv \text{ and } uSw\}$

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$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is an algebra of binary relations if there exists an equivalence relation E on some set A such that

- $\mathcal{R}$ is a set of subrelations of E that is closed under 0, $-$, $\cup$, Id, $^{-1}$, $\circ$

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is a relation algebra if

- $(\mathcal{R}, 0, -, \cup)$ is a Boolean algebra
- ${x^{-1}}^{-1} = x$
- $(x \cup y)^{-1} = x^{-1} \cup y^{-1}$
- $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$
- $(x \cup y) \circ z = (x \circ z) \cup (y \circ z)$
- $(x \circ y) \circ z = x \circ (y \circ z)$
- $x \circ Id = Id \circ x = x$
- $(x \circ y) \cap z = 0$ iff $(z \circ y^{-1}) \cap x = 0$ iff $(x^{-1} \circ z) \cap y = 0$

Tarski (1941)

- every algebra of binary relations is a relation algebra
- is every relation algebra isomorphic to an algebra of binary relations ?

Lyndon (1950)

- NO

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Proper and abstract fork algebras

On the origin of fork algebras

Let $\neq$ be

- $-Id$

Consider the formula

- $\forall u \, \forall v \, \forall w \, \exists a \, (a \neq u \wedge a \neq v \wedge a \neq w)$

Suppose

- $\mathcal{R}$ is a set of subrelations of an equivalence relation E on A that is closed under 0, $-$, $\cup$, Id, ${}^{-1}$, $\circ$
- $* : \; A \times A \rightarrow A$ is injective and surjective
- $\mathcal{R}$ is closed under $\underline{\nabla}$ where $R \underline{\nabla} S = \{(u, v * w) : \; uRv$ and $uSw\}$

Proper and abstract fork algebras

On the origin of fork algebras

The following are equivalent

- $\forall u\, \forall v\, \forall w\, \exists a\, (a \neq u \wedge a \neq v \wedge a \neq w)$
- $\forall u\, \forall v\, \forall w\, \exists a\, (a \neq u \wedge a\, (\neq \underline{\nabla} \neq)\, v * w)$
- $\forall u\, \forall v\, \forall w\, \exists a\, (u \neq^{-1} a \wedge a\, (\neq \underline{\nabla} \neq)\, v * w)$
- $\forall u\, \forall v\, \forall w\, (u\, (\neq^{-1} \circ (\neq \underline{\nabla} \neq))\, v * w)$
- $\forall u\, \forall v\, \forall w\, (u\, (\neq^{-1} \circ (\neq \underline{\nabla} \neq))\, v * w \leftrightarrow u\, 1\, v \wedge u\, 1\, w)$
- $\forall u\, \forall v\, \forall w\, (u\, (\neq^{-1} \circ (\neq \underline{\nabla} \neq))\, v * w \leftrightarrow u\, (1 \underline{\nabla} 1)\, v * w)$
- $(\neq^{-1} \circ (\neq \underline{\nabla} \neq)) = (1 \underline{\nabla} 1)$

Proper and abstract fork algebras

On the origin of fork algebras

Development of the classes of proper and abstract fork algebras

- Hæberer and Veloso (1991), Veloso *et al.* (1992): $u * v =$ the tree with subtrees u and v
- Veloso and Hæberer (1991): $u * v =$ concatenation of the finite strings u and v

Proper and abstract fork algebras

Definition of the classes

$(\mathcal{R}, A, 0, -, \cup, Id, ^{-1}, \circ, \underline{\nabla}, *)$ is a proper fork algebra if

- $\mathcal{R}$ is a set of subrelations of an equivalence relation E on A that is closed under 0, $-$, $\cup$, Id, $^{-1}$, $\circ$
- $*: \ A \times A \rightarrow A$ is injective
- $\mathcal{R}$ is closed under $\underline{\nabla}$ where $R \underline{\nabla} S = \{(u, v * w) : \ uRv \text{ and } uSw\}$

Proper and abstract fork algebras

Definition of the classes

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ, \underline{\nabla})$ is an abstract fork algebra if

- $(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is a relation algebra
- $x \underline{\nabla} y = (x \circ (Id \underline{\nabla} 1)) \cap (y \circ (1 \underline{\nabla} Id))$
- $(x \underline{\nabla} y) \circ (z \underline{\nabla} t)^{-1} = (x \circ z^{-1}) \cap (y \circ t^{-1})$
- $(Id \underline{\nabla} 1)^{-1} \underline{\nabla} (1 \underline{\nabla} Id)^{-1} \leq Id$

Cross is defined by the equation

- $x \otimes y ::= ((Id \underline{\nabla} 1)^{-1} \circ x) \underline{\nabla} ((1 \underline{\nabla} Id)^{-1} \circ y)$

Proper and abstract fork algebras

Definition of the classes

Theorem

- every proper fork algebra is an abstract fork algebra
- every abstract fork algebra is isomorphic to a proper fork algebra

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Separation and parallel composition

PRSPDL : syntax

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha;\beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

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Separation and parallel composition

PRSPDL : semantics

Models

- a model is a structure of the form $\mathcal{M} = (W, R, *, V)$ where
  - W is a nonempty set of states
  - R is a function $a \mapsto R(a) \subseteq W \times W$
  - $*$ is a ternary relation over W
  - V is a function $p \mapsto V(p) \subseteq W$

Truth conditions

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(p)^{\mathcal{M}} = V(p)$
  - $(\bot)^{\mathcal{M}}$ is empty
  - $(\neg\phi)^{\mathcal{M}} = W \setminus (\phi)^{\mathcal{M}}$
  - $(\phi \vee \psi)^{\mathcal{M}} = (\phi)^{\mathcal{M}} \cup (\psi)^{\mathcal{M}}$
  - $([\alpha]\phi)^{\mathcal{M}} = \{x\text{: for all } y \in W, \text{ if } x(\alpha)^{\mathcal{M}}y, \ y \in (\alpha)^{\mathcal{M}}\}$

Separation and parallel composition

PRSPDL : semantics

Truth conditions according to Benevides *et al.* (2011)

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(a)^{\mathcal{M}} = R(a)$
  - $(\phi?)^{\mathcal{M}} = \{(x, y)\colon x = y \text{ and } y \in (\phi)^{\mathcal{M}}\}$
  - $(s_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y * (x, z)\}$
  - $(s_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y * (z, x)\}$
  - $(r_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x * (y, z)\}$
  - $(r_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x * (z, y)\}$
  - $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x(\alpha)^{\mathcal{M}}z$ and $z(\beta)^{\mathcal{M}}y\}$
  - $(\alpha \cup \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \cup (\beta)^{\mathcal{M}}$
  - $(\alpha^{\star})^{\mathcal{M}} = \{(x, y)\colon \text{there exists } n \in \mathbb{N} \text{ and there exists}$ $z_0, \ldots, z_n \in W \text{ such that } x = z_0(\alpha)^{\mathcal{M}} \ldots (\alpha)^{\mathcal{M}} z_n = y\}$
  - $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z, t, u, v \in W \text{ such that}$ $x * (z, t),\ y * (u, v),\ z(\alpha)^{\mathcal{M}}u \text{ and } t(\beta)^{\mathcal{M}}v\}$

Separation and parallel composition

PRSPDL : semantics

Truth conditions according to Frias (2002)

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(a)^{\mathcal{M}} = R(a)$
  - $(\phi?)^{\mathcal{M}} = \{(x, y)\colon x = y \text{ and } y \in (\phi)^{\mathcal{M}}\}$
  - $(s_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y * (x, z)\}$
  - $(s_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y * (z, x)\}$
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  - $(r_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x * (z, y)\}$
  - $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x(\alpha)^{\mathcal{M}}z$ and $z(\beta)^{\mathcal{M}}y\}$
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  - $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z, t \in W \text{ such that}$ $y * (z, t),\ x(\alpha)^{\mathcal{M}}z \text{ and } x(\beta)^{\mathcal{M}}t\}$

Separation and parallel composition

PRSPDL : classes of models

A model $\mathcal{M} = (W, R, *, V)$ is said to be separated iff

- if $x * (y, z)$ and $x * (t, u)$, $y = t$ and $z = u$

A model $\mathcal{M} = (W, R, *, V)$ is said to be deterministic iff

- if $x * (z, t)$ and $y * (z, t)$, $x = y$

A model $\mathcal{M} = (W, R, *, V)$ is said to be serial iff

- $*(x, y)$ is nonempty

In a separated model $\mathcal{M} = (W, R, *, V)$ we have

- if $x(s_1)^{\mathcal{M}}z$ and $z(r_1)^{\mathcal{M}}y$, $x = y$
- if $x(s_2)^{\mathcal{M}}z$ and $z(r_2)^{\mathcal{M}}y$, $x = y$

In a deterministic separated model $\mathcal{M} = (W, R, *, V)$ we have

- if $x(r_1)^{\mathcal{M}}z$, $z(s_1)^{\mathcal{M}}y$, $x(r_2)^{\mathcal{M}}t$ and $t(s_2)^{\mathcal{M}}y$, $x = y$

Separation and parallel composition

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha;\beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

For all $i \in \{1,2\}$ and for all s_i-free programs α

- the programs s_i and α are not equally interpreted in all separated models

For all $i \in \{1,2\}$ and for all r_i-free programs α

- the programs r_i and α are not equally interpreted in all separated models

Separation and parallel composition

PRSPDL : expressivity

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

For all atomic programs a, b and for all $\parallel$-free programs α

- the programs $a \parallel b$ and α are not equally interpreted in all separated models

Separation and parallel composition

PRSPDL : expressivity

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

The following expressions are equally interpreted in all separated models for each programs α, β, for each formulas ϕ and for each atomic formulas p not occurring in α, β, ϕ

- $\langle \alpha \parallel \beta \rangle \phi$
- $\forall p \; (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))$

Separation and parallel composition

PRSPDL : a simple fragment

Restriction of the syntax

- $\alpha, \beta ::= a \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

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Separation and parallel composition

PRSPDL : a simple fragment

Axiomatization

- all tautologies modus ponens necessitation
- $[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$
- $\langle r_1\rangle\phi \to [r_1]\phi \quad \langle r_2\rangle\phi \to [r_2]\phi$
- $\phi \to [s_1]\langle r_1\rangle\phi \quad \phi \to [s_2]\langle r_2\rangle\phi \quad \phi \to [r_1]\langle s_1\rangle\phi \quad \phi \to [r_2]\langle s_2\rangle\phi$
- $\langle s_1\rangle\top \leftrightarrow \langle s_2\rangle\top \quad \langle r_1\rangle\top \leftrightarrow \langle r_2\rangle\top$
- $\langle s_1 ; r_1\rangle\phi \to [s_1 ; r_1]\phi \quad \langle s_2 ; r_2\rangle\phi \to [s_2 ; r_2]\phi$
- $[s_1 ; r_2]\phi \to \phi$
- $\phi \to [s_1 ; r_2]\langle s_1 ; r_2\rangle\phi$
- $[s_1 ; r_2]\phi \to [s_1 ; r_2][s_1 ; r_2]\phi$
- $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$
- $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid (\alpha; \beta) \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi \mid (\phi \circ \psi) \mid (\phi \triangleright \psi) \mid (\phi \triangleleft \psi)$

Semantics

- a model is a structure of the form $\mathcal{M} = (W, R, *, V)$ where
  - W is a nonempty set of states
  - R is a function $a \mapsto R(a) \subseteq W \times W$
  - $*$ is a ternary relation over W
  - V is a function $p \mapsto V(p) \subseteq W$

In a model $\mathcal{M} = (W, R, *, V)$ we define

- $(a)^{\mathcal{M}} = R(a)$
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- $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)$: there exists $z \in W$ such that $x(\alpha)^{\mathcal{M}} z$ and $z(\beta)^{\mathcal{M}} y\}$
- $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)$: there exists $z, t \in W$ such that $y * (z, t)$, $x(\alpha)^{\mathcal{M}} z$ and $x(\beta)^{\mathcal{M}} t\}$

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- $(p)^{\mathcal{M}} = V(p)$
- $(\bot)^{\mathcal{M}}$ is empty
- $(\neg\phi)^{\mathcal{M}} = W \setminus (\phi)^{\mathcal{M}}$
- $(\phi \vee \psi)^{\mathcal{M}} = (\phi)^{\mathcal{M}} \cup (\psi)^{\mathcal{M}}$
- $([\alpha]\phi)^{\mathcal{M}} = \{x$: for all $y \in W$, if $x(\alpha)^{\mathcal{M}}y$, $y \in (\alpha)^{\mathcal{M}}\}$
- $(\phi \circ \psi)^{\mathcal{M}} = \{x$: there exists $y, z \in W$ such that $x \in y \star z$, $y \in V_{\mathcal{M}}(\phi)$ and $z \in V_{\mathcal{M}}(\psi)\}$
- $(\phi \triangleright \psi)^{\mathcal{M}} = \{x$: there exists $y, z \in W$ such that $z \in y \star x$, $y \in V_{\mathcal{M}}(\phi)$ and $z \in V_{\mathcal{M}}(\psi)\}$
- $(\phi \triangleleft \psi)^{\mathcal{M}} = \{x$: there exists $y, z \in W$ such that $y \in x \star z$, $y \in V_{\mathcal{M}}(\phi)$ and $z \in V_{\mathcal{M}}(\psi)\}$

A model $\mathcal{M} = (W, R, *, V)$ is said to be separated iff

- if $x * (y, z)$ and $x * (t, u)$, $y = t$ and $z = u$

A model $\mathcal{M} = (W, R, *, V)$ is said to be deterministic iff

- if $x * (z, t)$ and $y * (z, t)$, $x = y$

A model $\mathcal{M} = (W, R, *, V)$ is said to be serial iff

- $*(x, y)$ is nonempty

PDL_0^Δ

PDL_0^Δ : modal definability

The elementary classes of frames defined by the first-order sentences in the hereunder table are modally definable by the associated formulas.

$\exists y \; y \in x \star x$	$\langle \top? \parallel \top? \rangle \top$
$(y \in x \star x \wedge z \in x \star x \to y = z)$	$\langle \top? \parallel \top? \rangle p \to [\top? \parallel \top?] p$
$(y \in x \star x \to x \in x \star y)$	$p \to [\top? \parallel \top?](p \triangleright p)$
$(y \in x \star x \to x \in y \star x)$	$p \to [\top? \parallel \top?](p \triangleleft p)$
$(z \in x \star y \leftrightarrow z \in y \star x)$	$p \circ q \leftrightarrow q \circ p$
$\exists y \; \exists z \; x \in y \star z$	$\top \circ \top$
$\exists y \; \exists z \; y \in z \star x$	$\top \triangleright \top$
$\exists y \; \exists z \; z \in x \star y$	$\top \triangleleft \top$
$(t \in (x \star y) \star z \leftrightarrow t \in x \star (y \star z))$	$(p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$
$x \notin y \star z$	$\bot \bar{\circ} \bot$

The class of all separated frames is modally definable by the fomula

- $p \circ q \rightarrow (p \bar{\circ} \bot) \wedge (\bot \bar{\circ} q)$

The class of all deterministic frames is not modally definable

For all test-free formulas ϕ

- the formulas $\langle \top? \parallel \top? \rangle \top$ and ϕ are not equally interpreted in all separated deterministic models

For all fork-free formulas ϕ

- the formulas $\langle a \parallel a \rangle \top$ and ϕ are not equally interpreted in all separated deterministic models

The following expressions are equally interpreted in all separated models for each programs α, β, for each formulas ϕ and for each atomic formulas *p* not occurring in α, β, ϕ

- $\langle \alpha \parallel \beta \rangle \phi$
- $\forall p\ (\langle \alpha \rangle ((\phi \wedge p) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\phi \wedge \neg p)))$

Axiomatization/completeness

PDL_0^{Δ} : axiomatization

Axioms

- all tautologies
- $[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$
- $\langle \alpha; \beta \rangle \phi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \phi$
- $\langle \alpha \parallel \beta \rangle \phi \to \langle \alpha \rangle ((\phi \wedge \psi) \triangleleft \top) \vee \langle \beta \rangle (\top \triangleright (\phi \wedge \neg \psi))$
- $\langle \phi ? \rangle \psi \leftrightarrow \phi \wedge \psi$
- $(\phi \to \psi) \bar{\circ} \chi \to (\phi \bar{\circ} \chi \to \psi \bar{\circ} \chi)$
- $\phi \bar{\circ} (\psi \to \chi) \to (\phi \bar{\circ} \psi \to \phi \bar{\circ} \chi)$
- $(\phi \to \psi) \bar{\triangleright} \chi \to (\phi \bar{\triangleright} \chi \to \psi \bar{\triangleright} \chi)$
- $\phi \bar{\triangleright} (\psi \to \chi) \to (\phi \bar{\triangleright} \psi \to \phi \bar{\triangleright} \chi)$
- $(\phi \to \psi) \bar{\triangleleft} \chi \to (\phi \bar{\triangleleft} \chi \to \psi \bar{\triangleleft} \chi)$
- $\phi \bar{\triangleleft} (\psi \to \chi) \to (\phi \bar{\triangleleft} \psi \to \phi \bar{\triangleleft} \chi)$

Axiomatization/completeness

$PDL_0^{\triangle}$: axiomatization

Axioms

- $\phi \circ \neg(\phi \triangleright \neg\psi) \to \psi$
- $\phi \triangleright \neg(\phi \circ \neg\psi) \to \psi$
- $\neg(\neg\phi \triangleleft \psi) \circ \psi \to \phi$
- $\neg(\neg\phi \circ \psi) \triangleleft \psi \to \phi$
- $[(\alpha; \phi?) \parallel (\beta; \psi?)](\phi \circ \psi)$
- $\langle\alpha(\phi?)\rangle\psi \to \langle\alpha((\phi \wedge \chi)?)\rangle\psi \vee \langle\alpha((\phi \wedge \neg\chi)?)\rangle\psi$
- $\langle f(\alpha)\rangle\phi \leftrightarrow \langle\alpha\rangle\phi$
- $p \circ q \to (p \bar{\circ} \bot) \wedge (\bot \bar{\circ} q)$

Axiomatization/completeness

PDL_0^{Δ} : axiomatization

Inference rules

- modus ponens : from ϕ and $\phi \rightarrow \psi$, infer ψ
- necessitation : from ϕ, infer $[\alpha]\phi$, $\phi \bar{\circ} \psi$ and $\psi \bar{\circ} \phi$
- fork : from $\{[\gamma](\langle\alpha\rangle((\phi \wedge p) \lhd \top) \vee \langle\beta\rangle(\top \rhd (\phi \wedge \neg p))) :$ p is a propositional variable$\}$, infer $[\gamma](\langle\alpha \parallel \beta\rangle\phi)$

Axiomatization/completeness

PDL_0^{Δ} : completeness

...

Open problems

Truth conditions of Benevides *et al.* (2011)

- Decidability/complexity of satisfiability for the restriction considered by Benevides *et al.* (2011)
- Decidability/complexity of satisfiability for the full language
- Tableau calculus for the restriction considered by Benevides *et al.* (2011)
- Tableau calculus for the full language
- Axiomatization of validity for the full language

Truth conditions of Frias (2002)

- Same issues

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