Refinement Modal Logic

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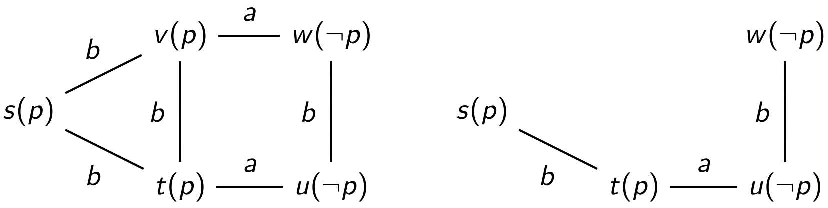
Fine *Propositional quantifiers in modal logic* Theoria 1970

Kit Fine distinguishes three ways of quantification:

1. quantification over boolean definable subsets;
2. quantification over modally definable subsets;
3. quantification over any subset.



1. **Boolean**: $\{s, t, u, v, w\}$, $\{s, t, v\}$, and $\{u, w\}$.
2. **Modally**: also $\{s\}$ (by $\Box_a p$) and $\{t, v\}$ (by $p \wedge \neg\Box_a p$).
3. **Any**: E.g., $\{v\}$ is not modally definable. In the restriction excluding v, w has become modally different from state u.

Different ways of quantifying over information change

- there is an announcement after which φ;
- there is an announcement by the agents in group G after which φ;
- there is an announcement by the agents in group G after which, no matter what the remaining agents announce, φ;
- there is a refinement of the accessibility relation after which φ;
- there is an action model after which φ;
- there is an action model with precondition ψ after which φ;
- there is a modal refinement after which φ;
- there is a model minus a state different from the actual state after which φ (sabotage logic);
- there is . . . any other submodel operation after which φ.

Motivation — philosophical logic, epistemic planning

philosophical logic

- **'Fitch's knowability'**: If φ is true, φ is knowable.
- *Knowable formulas*: such that $\varphi \rightarrow \exists\Box\varphi$
  If φ is true, there is an announcement after which φ is known.

epistemic planning

- **synthesis/planning**
  You have an information state, and a goal information state; can we reach the goal by an action sequence?
- **dynamic epistemic specifications**
  The DEL equivalent for **temporal epistemic specifications***: quantifying over information change*

Think of dynamic epistemic quantifiers as temporal modalities:
$\Box_a p \rightarrow \forall\Box_b(\varphi \wedge \exists\Box^*_{ab}\psi)$
$\Box_a p \rightarrow G\Box_b(\varphi \wedge F\Box^*_{ab}\psi)$

Refinement Modal Logic

- In arbitrary public announcement logic (APAL) we quantify over announcements: modally definable subsets (denotations of quantifier free formulas).
- In arbitrary action model logic (AAML) we quantify over action models (with quantifier free preconditions).
- In these logics the quantification is over dynamic modalities for action execution . . .
- . . . but alternatively it is over model restrictions that are denotations of formulas.
- What about a new trick?
- We now propose a new form of quantification, independent from the logical language.
- It is called refinement quantification, or just refinement.
- Refinement is the dual of simulation. There are lots of results in computer science for simulation.

Refinement Modal Logic — What is a refinement?

Consider this pointed model (epistemic state) M:

M' is a bisimilar copy of the model M:

M'' is a submodel (model restriction) of M:

M''' is a refinement of M: (M is a simulation of M''':)

A refinement of a model is a submodel of a bisimilar model:

Refinement Modal Logic

Refinement:

- $\underline{\leftrightarrow}$ bisimulation: atoms, forth, back
- $\underline{\rightarrow}$ simulation: atoms, forth
- $\underline{\leftarrow}$ refinement: atoms, back

Refinement for agent a:

- for agent a: atoms, back
- for all other agents: atoms, forth, back

Refinement for group of agents B:

- for agents $a \in B$: atoms, back
- for all other agents: atoms, forth, back

Refinement Modal Logic — bisimulation & refinement

Let two models $M = (S, R, V)$ and $M' = (S', R', V')$ be given.
A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a bisimulation if for all $(s, s') \in \mathfrak{R}$, $a \in A$, $p \in P$:

- **atoms** $s \in V(p)$ iff $s' \in V'(p)$;
- **forth-a** if $R_a st$, there is a t' such that $R'_a s' t'$ and $(t, t') \in \mathfrak{R}$;
- **back-a** if $R'_a s' t'$, there is a t such that $R_a st$ and $(t, t') \in \mathfrak{R}$.

We write $(M, s) \leftrightarroweq_A (M', s')$. *(or just $(M, s) \leftrightarroweq (M', s')$)*

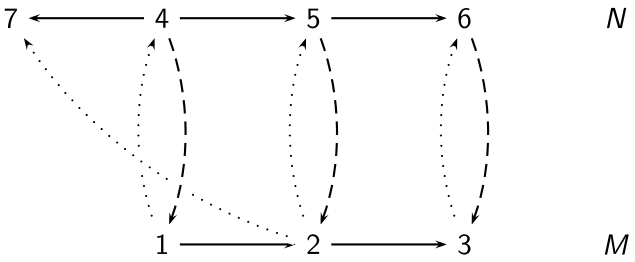
A relation $\mathfrak{R}_B$ that satisfies **atoms**, **back**-a, and **forth**-a for every $a \in A \setminus B$, and that satisfies **atoms**, and **back**-b for every $b \in B$, is a *B-refinement*.

We write $(M, s) \succeq_B (M', s')$.

Refinement Modal Logic — refinement

- $\succeq_a$ is reflexive and transitive (a preorder), and satisfies Church-Rosser;
- $(M,s)(\succeq_{a_1} \circ \cdots \circ \succeq_{a_n})(M,t)$ iff $(M,s)\succeq_{\{a_1,\dots,a_n\}}(M,t)$;
- we may have $(N,t)\succeq_a(M,s)$ and $(M,s)\succeq_a(N,t)$ but not $(M,s)\leftrightarrows_a(N,t)$.

Refinement and simulation, but no bisimulation:



Refinement is bisimulation plus model restriction

Given a pointed model, choose a bisimilar pointed model, then remove some pairs from the accessibility relation for a.

Given a pointed model, choose a bisimilar pointed model except for variable q, with q (only) false in some states that are accessible for a, then remove all those pairs from the accessibility relation for a.

Given a pointed model, choose a bisimilar pointed model except for variable q, then remove all pairs from the accessibility relation for a pointing to states where q is false.

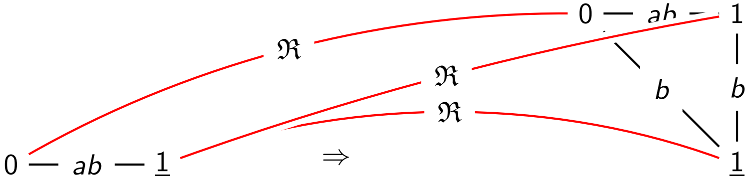
Given a pointed model, choose a bisimilar pointed model except for variable q, then restrict the model to the states where q is true.

Proposition: Given $(M, s) \underline{\leftarrow}_a (N, t)$, there is a (N', t) and a $p \in P$ such that $(M, s) \underline{\leftrightarrow}^{\overline{p}} (N', t)$ and $(N', t)|p = (N, t)$.

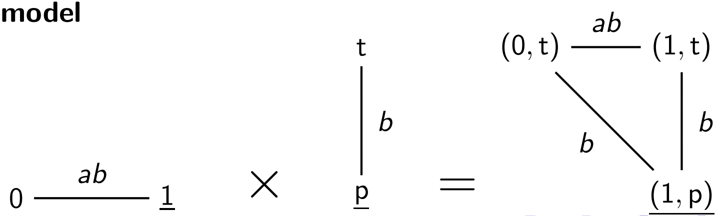
Refinement and action models

Two agents a, b are uncertain about the value of a (true) fact p. An informative event is possible after which a knows that p but b does not know that.

Refinement



Action model



Refinement and action models

Proposition Action model execution is a refinement. A refinement of a *finite* epistemic model is the result of action model execution.

Sketch of proof
$\Rightarrow$ Given pointed model (M, s) and epistemic action (M, s), the resulting $(M \otimes \mathsf{M}, (s, \mathsf{s}))$ is a refinement of (M, s) by way the relation $\mathfrak{R}$ consisting of all pairs $(t, (t, \mathsf{t}))$ such that $(M, t) \models \mathsf{pre}(\mathsf{t})$.
$\Leftarrow$ Consider epistemic action (M, s') that is isomorphic to a given refinement (N, s') of a model (M, s), but wherein valuations in states $t \in N$ are replaced by preconditions. Precondition $\mathsf{pre}(t)$ should be satisfied in exactly the states $s \in M$ such that $(s, t) \in \mathfrak{R}$, where $\mathfrak{R} : (M, s) \underline{\leftarrow}_A (N, s')$. In a *finite* model, we can single out these states by a distinguishing formula. $(M \otimes \mathsf{M}, (s, s'))$ can be bisimulation contracted to (N, s').

Refinement Modal Logic — language and semantics

Language $\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid \forall_a\varphi$

Structures pointed Kripke models

Semantics

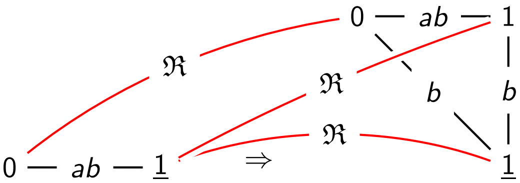
$(M, s) \models p$	iff	$s \in V_p$
$(M, s) \models \neg\varphi$	iff	$(M, s) \not\models \varphi$
$(M, s) \models \varphi \wedge \psi$	iff	$(M, s) \models \varphi$ and $(M, s) \models \psi$
$(M, s) \models \Box_a\varphi$	iff	for all $t \in S : R_a st$ implies $(M, t) \models \varphi$
$(M, s) \models \forall_a\varphi$	iff	$\forall(M', s') : (M, s) \succeq_a (M', s')$ implies $(M', s') \models \varphi$

Dual:
$(M, s) \models \exists_a\varphi$ iff $\exists(M', s') : (M, s) \succeq_a (M', s')$ implies $(M', s') \models \varphi$

[Bozzelli, van Ditmarsch, French, Hales, Pinchinat,
Refinement Modal Logic, manuscript (ArXiV).]

Refinement Modal Logic – example

Two agents a, b are uncertain about the value of a (true) fact p. An informative event is possible after which a knows that p but b does not know that.



$\exists_a(\Box_a p \land \neg\Box_b\Box_a p)$ $\qquad$ $\Box_a p \land \neg\Box_b\Box_a p$

Refinement Modal Logic — validities

1. $\models \exists_a \exists_b \varphi \leftrightarrow \exists_b \exists_a \varphi$.
2. $\models \forall_a \varphi \rightarrow \varphi$ (reflexivity)
3. $\models \forall_a \varphi \rightarrow \forall_a \forall_a \varphi$ (transitivity)
4. $\models \exists_a \forall_a \varphi \rightarrow \forall_a \exists_a \varphi$ (Church-Rosser)
5. $\models \exists_a \Diamond_a \varphi \leftrightarrow \Diamond_a \exists_a \varphi$

Proof of 5.

$\Rightarrow$: Given $(M, s) \models \exists_a \Diamond_a \varphi$, and (M', s') s.t. $(M, s) \succeq_a (M', s')$ and $t' \in s' R'_a$; then $(M', s') \models \Diamond_a \varphi$, and $(M', t') \models \varphi$. Because of **back**, there is a $t \in sR_a$ such that $(M, t) \succeq_a (M', t')$. Therefore $(M, t) \models \exists_a \varphi$ and thus $(M, s) \models \Diamond_a \exists_a \varphi$.

$\Leftarrow$: Given $(M, s) \models \Diamond_a \exists_a \varphi$, and $t \in sR_a$ and (M', t') such that $(M, t) \models \exists_a \varphi$ and $(M', t') \models \varphi$. Consider the model N with point s that is the disjoint union of M and M' except that: all outgoing a-arrows from s in M are removed (all pairs $(s, t) \in R_a$), a new a-arrow links s to t' in M' (add (s, t') to the new R_a). Then (N, s) is an a-refinement of (M, s) that, obviously, satisfies $\Diamond_a \varphi$, so (M, s) satisfies $\exists_a \Diamond_a \varphi$.

Refinement Modal Logic

We had: *Refinement is bisimulation plus model restriction.*
Analogously we have: *Refinement quantification is bisimulation quantification plus relativization.* This requires the notion of agent relativization.

- Define agent relativization $\varphi^{(a,p)}$ on formulas in bisimulation quantified modal logic with crucial clauses $(\Box_a\varphi)^{(a,p)} = \Box_a(p \rightarrow \varphi^{(a,p)})$ and $(\Box_b\varphi)^{(a,p)} = \Box_b\varphi^{(a,p)}$, and $(\tilde{\forall} p\varphi)^{(a,p)} = \tilde{\forall} q\varphi[q\backslash p]^{(a,p)}$ (q not in φ);
- Prove that $(\varphi^{(a,p)})^{(b,q)} = (\varphi^{(b,q)})^{(a,p)}$;
- Define a translation from refinement modal logic to bisimulation quantified modal logic with crucial clause $t(\forall_a\varphi) = \tilde{\forall} p\ t(\varphi)^{(a,p)}$ (p not in φ);
- Every formula in refinement modal logic is logically equivalent to its translation in bisimulation quantified modal logic, pregnant examples (for $\forall$-free φ):
- $\exists_a\varphi$ is equivalent to $\tilde{\exists} p\varphi^{(a,p)}$;
- $\exists\varphi$ is equivalent to $\tilde{\exists} p\varphi^p$.

Refinement Modal Logic — relativization example

The quantifier $\tilde{\exists} p$ is a bisimulation quantifier. The quantifier $\exists_a$ is the simulation quantifier; given $\exists_a \varphi$, $\exists_a$ implicitly quantifies over a variable q not occurring in φ.

$$
\begin{array}{lclcl}
t(\exists_a \exists_b r) & = & \tilde{\exists} p \; t(\exists_b r)^{(a,p)} & = & \\
\tilde{\exists} p (\tilde{\exists} p \; t(r)^{(b,p)})^{(a,p)} & = & \tilde{\exists} p (\tilde{\exists} p \; r^{(b,p)})^{(a,p)} & = & \\
\tilde{\exists} p (\tilde{\exists} p \; r)^{(a,p)} & = & \tilde{\exists} p \tilde{\exists} q \; r^{(a,q)} & = & \tilde{\exists} p \tilde{\exists} q \; r
\end{array}
$$

Refinement Modal Logic — Axiomatization

We first introduce the cover operator for modal logic.

- $\nabla_a \Phi$ abbreviates $\Box_a \bigvee_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi} \Diamond_a \varphi$
  $\bigvee_{\varphi \in \emptyset} \varphi$ is always false, whilst $\bigwedge_{\varphi \in \emptyset} \varphi$ is always true.
- Allow abbreviations so $\nabla_a \Phi$ is $\Box_a \bigvee \Phi \wedge \bigwedge \Diamond_a \Phi$.
- $\Box_a \varphi$ iff $\nabla_a \emptyset \vee \nabla_a \{\varphi\}$, and $\Diamond_a \varphi$ iff $\nabla_a \{\varphi, \top\}$.
- Conjunction of two cover formula is again a cover formula
  $\nabla_a \Phi \wedge \nabla_a \Psi \quad \Leftrightarrow \quad \nabla_a (\Phi \wedge \bigvee \Psi \ \cup \ \bigvee \Phi \wedge \Psi)$.
- Every formula in multi-agent modal logic is equivalent to a disjunctive form $\varphi ::= (\varphi \vee \varphi) \mid (\varphi_0 \wedge \bigwedge_{a \in B} \nabla_a \{\varphi, \dots, \varphi\})$ (where φ_0 is a propositional formula). No negations in front of modalities!

Try to forget this, and instead look at the axiomatization.

Refinement Modal Logic — Axiomatization

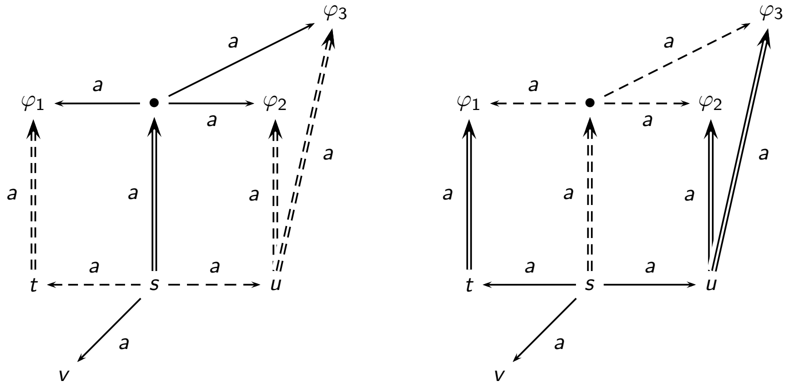
Prop	All tautologies of propositional logic	
K	$\Box_a(\varphi \to \psi) \to \Box_a\varphi \to \Box_a\psi$	
R	$\forall_a(\varphi \to \psi) \to \forall_a\varphi \to \forall_a\psi$	
RProp	$\forall_a p \leftrightarrow p$ and $\forall_a \neg p \leftrightarrow \neg p$	
RK	$\exists_a \nabla_a \Phi \leftrightarrow \bigwedge \Diamond_a \exists_a \Phi$	
RKmulti	$\exists_a \nabla_b \Phi \leftrightarrow \nabla_b \exists_a \Phi$	where $a \neq b$
RKconj	$\exists_a \bigwedge_{b\in B} \nabla_b \Phi^b \leftrightarrow \bigwedge_{b\in B} \exists_a \nabla_b \Phi^b$	
MP	From $\varphi \to \psi$ and φ infer ψ	
NecK	From φ infer $\Box_a\varphi$	
NecR	From φ infer $\forall_a\varphi$	

Completeness: This is a standard reduction argument; refinement modal logic is equally expressive as multi-agent modal logic. We note the disjunctive form of the interaction axioms for refinement quantifiers and modalities. The reduction argument is by induction on the structure of disjunctive forms.

Refinement Modal Logic — Axiom RK

Interaction between refinement and modality in axiom **RK**.

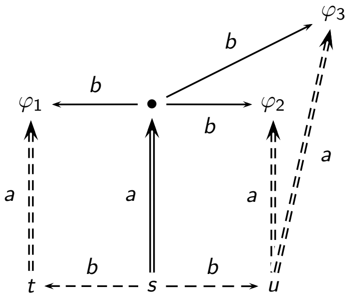
$$\mathbf{RK} \quad \exists_a \nabla_a \Phi \leftrightarrow \bigwedge \Diamond_a \exists_a \Phi$$

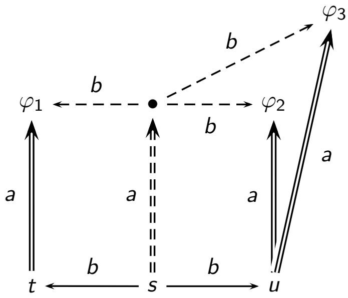


Refinement Modal Logic — Axiom RKmulti

Interaction between refinement and modality in axiom **RKmulti**.

RKmulti $\exists_a \nabla_b \Phi \leftrightarrow \nabla_b \exists_a \Phi$ where $a \neq b$





Refinement Modal Logic — Instantiation of RK

How **RK** axiom works as a reduction principle for $\exists\Box\varphi$ and $\exists\Diamond\varphi$.

$$\begin{array}{rcl}\exists\Box\varphi & \leftrightarrow & \exists(\nabla\{\varphi\} \vee \nabla\emptyset) \\ & \leftrightarrow & \exists\nabla\{\varphi\} \vee \exists\nabla\emptyset \\ & \leftrightarrow & \exists\nabla\{\varphi\} \vee \exists\Box\bot \\ & \leftrightarrow & \top\end{array}$$

and

$$\begin{array}{rcl}\exists\Diamond\varphi & \leftrightarrow & \exists\nabla\{\varphi, \top\} \\ & \leftrightarrow & \Diamond\exists\varphi \wedge \Diamond\exists\top \\ & \leftrightarrow & \Diamond\exists\varphi\end{array}$$

$\exists\Box\varphi \leftrightarrow \top$ and $\exists\Diamond\varphi \leftrightarrow \Diamond\exists\varphi$ are not axioms instead of **RK** as the axiomatization would not be complete. **RK** is much more powerful as this allows any finite Φ.

Refinement Modal Logic — Example derivation

We show $\vdash (\Diamond_a p \wedge \Diamond_b p \wedge \Diamond_a \neg p \wedge \Diamond_b \neg p) \rightarrow \exists_a (\Box_a p \wedge \neg \Box_b p)$. Let φ be $(\Diamond_a p \wedge \Diamond_b p \wedge \Diamond_a \neg p \wedge \Diamond_b \neg p)$.

$\vdash \varphi \rightarrow \Diamond_a p \wedge \Diamond_b \neg p$	**Prop**
$\vdash \varphi \rightarrow \Diamond_a p \wedge \nabla_b \{\neg p, \top\}$	Definition of ∇
$\vdash \varphi \rightarrow \Diamond_a \neg\neg p \wedge \nabla_b \{\neg\neg\neg p, \neg\neg\top\}$	**Prop**
$\vdash \varphi \rightarrow \Diamond_a \neg\forall_a \neg p \wedge \nabla_b \{\neg\forall_a \neg\neg p, \neg\forall_a \neg\top\}$	**RProp**
$\vdash \varphi \rightarrow \Diamond_a \exists_a p \wedge \nabla_b \{\exists_a \neg p, \exists_a \top\}$	Definition of $\exists$
$\vdash \varphi \rightarrow \exists_a \nabla_a \{p\} \wedge \nabla_b \{\exists_a \neg p, \exists_a \top\}$	**RK**
$\vdash \varphi \rightarrow \exists_a \nabla_a \{p\} \wedge \exists_a \nabla_b \{\neg p, \top\}$	**RKmulti**
$\vdash \varphi \rightarrow \exists_a (\nabla_a \{p\} \wedge \nabla_b \{\neg p, \top\})$	**RKconj**
$\vdash \varphi \rightarrow \exists_a (\Box_a p \wedge \Diamond_b \neg p)$	Definition of ∇
$\vdash \varphi \rightarrow \exists_a (\Box_a p \wedge \neg \Box_b p)$	Definition of $\Diamond$

Refinement Epistemic Logic

How about quantifying over **information** change? Finally, the downside: S5 is harder than K! The interpretation of $\exists$ and $\forall$ is different. Also, (therefore,) the axiomatization refinement epistemic logic ($S5$) is not an extension of the axiomatization of refinement modal logic.

The semantic interpretation of $\forall$ over model class $\mathcal{C}$ is:

$$M_s \models \forall_a \varphi \text{ iff for all } M'_{s'} \in \mathcal{C} : \; M_s \succeq_a M'_{s'} \text{ implies } M'_{s'} \models \varphi.$$

$\exists\Box\bot$ is a validity in RML, but not in refinement epistemic logic (seriality must be preserved).

Refinement Epistemic Logic — Axiomatization

$$\textbf{RK} \quad \exists\nabla\Phi \leftrightarrow \bigwedge \Diamond\exists\Phi$$

Axiom **RK** is invalid for refinement epistemic logic. In $\mathcal{S}5$, $\exists\nabla(K p, \neg K p)$ is inconsistent, but $\Diamond\exists K p \wedge \Diamond\exists\neg K p$ is consistent: you do not consider an informative development possible after which you both know and don't know p at the same time. Instead:

$$\textbf{RS5} \quad \exists\nabla\Phi \leftrightarrow (\bigvee \Phi \wedge \bigwedge \Diamond\Phi),$$

$$\textbf{RKD45} \quad \exists\nabla\Phi \leftrightarrow \bigwedge \Diamond\Phi,$$

(Where Φ is a set of *propositional* formulas.) **RS5** instead of **RK**, plus the usual **S5** axioms **T**, **4**, and **5**, is a complete axiomatization for the refinement epistemic logic. For *Refinement doxastic logic*, add axioms **D** (for seriality), **4**, and **5** and **RKD45** to get a complete axiomatization.

Refinement Modal Logic — Theory, further research

- refinement modal logic is decidable . . . *and this is somewhat surprising, as arbitrary action model logic may be undecidable (the matter has not yet been decided)*;
- complexity of satisfiability, single-agent RML: in between single and double exponential (see JELIA 2012);
- extension of the language of refinement modal logic with fixed points: refinement μ-calculus;
- single-agent refinement doxastic logic (M4M Osuna 2011), multi-agent refinement epistemic logic (AiML 12); ...
- future investigations: refinement CTL, refinement PDL, ...

- Laura Bozzelli, Hans van Ditmarsch, Sophie Pinchinat. *Complexity of one-agent Refinement Modal Logic.* JELIA.
- James Hales, Tim French and Rowan Davies. *Refinement Quantified Logics of Knowledge and Belief For Multiple Agents.* AiML 2012.
- Laura Bozzelli, Hans van Ditmarsch, Tim French, James Hales, Sophie Pinchinat. *Refinement Modal Logic.* http://arxiv.org/abs/1202.3538
- J. Hales. *Refinement quantifiers for logics of belief and knowledge.* Honours Thesis, Univ. of Western Australia, 2011.
- J. Hales, T. French, and R. Davies. *Refinement quantified logics of knowledge.* ENTCS, 278:85-98, 2011.
- Hans van Ditmarsch, Tim French, and Sophie Pinchinat. *Future Event Logic - axioms and complexity.* AiML, Volume 8, pages 77-99, 2010.
- H.P. van Ditmarsch and T. French. *Simulation and information.* LNAI 5605, pages 51-65, 2009.

Only the beginning ... Different forms of quantifying ...

- there is an announcement after which φ;
- there is an announcement by the agents in group G after which φ;
- there is an announcement by the agents in group G after which, no matter what the remaining agents announce, φ;
- there is a refinement of the accessibility relation after which φ;
- there is an action model after which φ;
- there is an action model with precondition ψ after which φ;
- there is a modal refinement after which φ;
- there is a model minus a state different from the actual state after which φ (sabotage logic);
- there is ... any other submodel operation after which φ.

Thank you!