A short introduction to propositional dynamic logic with separation and parallel composition

CNRS - INPT - UPS - UT1 - UTM

Contents

Specification languages must allow for a modular description of

- structural properties
- dynamic properties
- temporal properties

Different formalisms allow us to specify these properties

- first-order classical logic
- propositional and first-order dynamic logic
- different modal logics

An amalgamating formalism should

- be expressive enough
- have very simple semantics
- have a complete and simple deductive system

The formalism called fork algebras was proposed to this end

- it is presented in the form of an equational calculus
- it is complete with respect to a very simple semantics

Algebras of binary relations on some set A

- 0, empty binary relation
- $-R$, complement of a binary relation R with respect to a largest relation E
- $R \cup S$, union of binary relations R and S
- Id, identity binary relation on A
- R^{-1}, transposition of a binary relation R
- $R \circ S$, composition of binary relations R and S

Monk (1964)

- a class of agebras containing these operations cannot be axiomatized by a finite set of equations

Bibliography

Monk, J.: *On representable relation algebras.* Michigan Mathematical Journal **11** (1964) 207–210.

In order to overcome this drawback

- an extra binary operation on relations called fork is added

Addition of fork has two main consequences

- the class of algebras obtained can be axiomatized by a finite set of equations
- it induces a structure on the domain on top of which relations are built

Introduction and motivations

Software specification, binary relations and fork

Algebras of binary relations on some set A closed under a binary function $\star$

- $R \underline{\triangledown} S$, fork of binary relations R and S

The definition of the operation fork is given by

- $R \underline{\triangledown} S = \{(x, y \star z) : \ xRy \text{ and } xSz\}$

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Frias, M., Baum, G., Hæberer, A., Veloso, P.: *Fork algebras are representable.* Bulletin of the Section of Logic **24** (1995) 64–75.

Frias, M., Hæberer, A., Veloso, P.: *A finite axiomatization for fork algebras.* Logic Journal of the IGPL **5** (1997) 311–319.

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is an algebra of binary relations if

- E, binary relation on a set A
- $\mathcal{R}$, set of binary relations on A
- if $R \in \mathcal{R}$, $R \subseteq E$
- $\mathcal{R}$ is closed under 0, $-$, $\cup$, Id, $^{-1}$, $\circ$

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is full if moreover

- its universe is of the form $2^{U \times U}$ for some set U

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is square if moreover

- its largest relation is of the form $U \times U$ for some set U

Theorem

- every full algebra of binary relations is square
- a square algebra of binary relations whose largest relation is $U \times U$ is a subalgebra of the full algebra of binary relations with universe $2^{U \times U}$
- every algebra of binary relations is embeddable in a direct product of full algebras of binary relations

Elementary theory of binary relations: Tarski (1941)

- syntax
  - $R, S ::= P \mid 0 \mid -R \mid (R + S) \mid 1' \mid R^{-1} \mid (R; S)$
  - $\phi, \psi ::= R = S \mid xRy \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \forall x\ \phi$
- axiomatization
  - $\forall x\ \forall y\ \neg x0y$
  - $\forall x\ \forall y\ (x - Ry \leftrightarrow \neg xRy)$
  - $\forall x\ \forall y\ (x(R + S)y \leftrightarrow xRy \vee xSy)$
  - $\forall x\ x1'x$
  - $\forall x\ \forall y\ \forall z\ (xRy \wedge y1'z \rightarrow xRz)$
  - $\forall x\ \forall y\ (xR^{-1}y \leftrightarrow yRx)$
  - $\forall x\ \forall y\ (x(R; S)y \leftrightarrow \exists z\ (xRz \wedge zSy))$
  - $R = S \leftrightarrow \forall x\ \forall y\ (xRy \leftrightarrow xSy)$

Calculus of relations: Tarski (1941)

- syntax
  - $R, S ::= P \mid 0 \mid -R \mid (R + S) \mid 1' \mid R^{-1} \mid (R; S)$
  - $\phi, \psi ::= R = S \mid \perp \mid \neg\phi \mid (\phi \vee \psi)$
- axiomatization
  - axiomatization for Boolean algebras
  - ${R^{-1}}^{-1} = R$
  - $(R; S)^{-1} = S^{-1}; R^{-1}$
  - $(R; S); T = R; (S; T)$
  - $R; 1' = R$
  - $(R; S) \cdot T^{-1} = 0 \rightarrow (S; T) \cdot R^{-1} = 0$
  - $R; 1 = 1 \vee 1; -R = 1$

$(A, 0, -, \cup, Id, ^{-1}, \circ)$ is a relation algebra if

- $(A, 0, -, \cup)$ is a Boolean algebra
- ${x^{-1}}^{-1} = x$
- $(x \cup y)^{-1} = x^{-1} \cup y^{-1}$
- $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$
- $(x \cup y) \circ z = (x \circ z) \cup (y \circ z)$
- $(x \circ y) \circ z = x \circ (y \circ z)$
- $x \circ Id = Id \circ x = x$
- $(x \circ y) \cap z = 0$ iff $(z \circ y^{-1}) \cap x = 0$ iff $(x^{-1} \circ z) \cap y = 0$

Theorem

- every algebra of binary relations is a relation algebra

Questions: Tarski (1941)

- is every model of the calculus of relations isomorphic to an algebra of binary relations
- is it true that every formula of the calculus of relations that is valid in all algebras of binary relations is provable in the calculus of relations
- is it true that every formula of the elementary theory of binary relations can be transformed into an equivalent formula of the calculus of relations

Answers to Tarski's questions

- is every model of the calculus of relations isomorphic to an algebra of binary relations: NO, Lyndon (1950, 1956) and McKenzie (1970)
- is it true that every formula of the calculus of relations that is valid in all algebras of binary relations is provable in the calculus of relations: NO, Lyndon (1950)
- is it true that every formula of the elementary theory of binary relations can be transformed into an equivalent formula of the calculus of relations: NO, Tarski *et al.* (1987)

Proper and abstract fork algebras

On the origin of fork algebras

Recall the formula

- $\forall x\ \forall y\ \forall z\ \exists u\ (u0'x \wedge u0'y \wedge u0'z)$

Suppose we have some binary operator $\underline{\triangledown}$ and some binary function $\star$ such that

- $R \underline{\triangledown} S = \{(x, y \star z) :\ xRy \text{ and } xSz\}$

Proper and abstract fork algebras

On the origin of fork algebras

The following are equivalent

- $\forall x\, \forall y\, \forall z\, \exists u\, (u\, 0'\, x \;\wedge\; u\, 0'\, y \;\wedge\; u\, 0'\, z)$
- $\forall x\, \forall y\, \forall z\, \exists u\, (u\, 0'\, x \;\wedge\; u\, (0' \underline{\nabla} 0')\, y \star z)$
- $\forall x\, \forall y\, \forall z\, \exists u\, (x\, 0'^{-1}\, u \;\wedge\; u\, (0' \underline{\nabla} 0')\, y \star z)$
- $\forall x\, \forall y\, \forall z\, (x\, (0'^{-1} \circ (0' \underline{\nabla} 0'))\, y \star z)$
- $\forall x\, \forall y\, \forall z\, (x\, (0'^{-1} \circ (0' \underline{\nabla} 0'))\, y \star z \;\leftrightarrow\; x\, 1\, y \;\wedge\; x\, 1\, z)$
- $\forall x\, \forall y\, \forall z\, (x\, (0'^{-1} \circ (0' \underline{\nabla} 0'))\, y \star z \;\leftrightarrow\; x\, (1 \underline{\nabla} 1)\, y \star z)$
- $(0'^{-1} \circ (0' \underline{\nabla} 0')) = (1 \underline{\nabla} 1)$

Development of the classes of proper and abstract fork algebras

- Hæberer and Veloso (1991), Veloso *et al.* (1992): $x \star y =$ the tree with subtrees x and y
- Veloso and Hæberer (1991): $x \star y =$ concatenation of the finite strings x and y
- Mikulás *et al.* (1992): the class of all algebras with binary relations with an operator $\underline{\triangledown}$ defined by $R \underline{\triangledown} S = \{(x, y \star z) : \; xRy \text{ and } xSz\}$ is not finitely axiomatizable

Proper and abstract fork algebras

Definition of the classes

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ, \underline{\triangledown}, \star)$ is a star proper fork algebra if

- $(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ)$ is an algebra of binary relations on some set A
- $\star : \ A \times A \to A$ is injective
- $\mathcal{R}$ is closed under $\underline{\triangledown}$ where $R \underline{\triangledown} S = \{(x, y \star z) : \ xRy \text{ and } xSz\}$

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ, \underline{\triangledown}, \star)$ is full if moreover

- its universe is of the form $2^{U \times U}$ for some set U

$(\mathcal{R}, 0, -, \cup, Id, ^{-1}, \circ, \underline{\triangledown}, \star)$ is square if moreover

- its largest relation is of the form $U \times U$ for some set U

Proper and abstract fork algebras

Definition of the classes

$(A, 0, -, \cup, Id, ^{-1}, \circ, \underline{\nabla}, \star)$ is an abstract fork algebra if

- $(A, 0, -, \cup, Id, ^{-1}, \circ)$ is a relation algebra
- $x \underline{\nabla} y = (x \circ (Id \underline{\nabla} 1)) \cap (y \circ (1 \underline{\nabla} Id))$
- $(x \underline{\nabla} y) \circ (z \underline{\nabla} t)^{-1} = (x \circ z^{-1}) \cap (y \circ t^{-1})$
- $(Id \underline{\nabla} 1)^{-1} \underline{\nabla} (1 \underline{\nabla} Id)^{-1} \leq Id$

Cross is defined by the equation

- $x \otimes y ::= ((Id \underline{\nabla} 1)^{-1} \circ x) \underline{\nabla} ((1 \underline{\nabla} Id)^{-1} \circ y)$

Theorem

- every full proper fork algebra is square
- a square proper fork algebra whose largest relation is $U \times U$ is a subalgebra of the full proper fork algebra with universe $2^{U \times U}$
- every proper fork algebra is embeddable in a direct product of full proper fork algebras

Theorem

- every proper fork algebra is an abstract fork algebra
- every abstract fork algebra is isomorphic to a proper fork algebra

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Separation and parallel composition

PRSPDL

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Semantics

- a model is a structure of the form $\mathcal{M} = (W, R, *, V)$ where
  - W is a nonempty set of states
  - R is a function $a \mapsto R(a) \subseteq W \times W$
  - $*$ is a ternary relation over W
  - V is a function $p \mapsto V(p) \subseteq W$

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Frias, M.: *Fork Algebras in Algebra, Logic and Computer Science.* World Scientific (2002).

Separation and parallel composition

Truth conditions according to Benevides *et al.* (2011)

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(a)^{\mathcal{M}} = R(a)$
  - $(\phi?)^{\mathcal{M}} = \{(x, y)\colon x = y \text{ and } y \in (\phi)^{\mathcal{M}}\}$
  - $(s_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y \star (x, z)\}$
  - $(s_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } y \star (z, x)\}$
  - $(r_1)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x \star (y, z)\}$
  - $(r_2)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x \star (z, y)\}$
  - $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z \in W \text{ such that } x(\alpha)^{\mathcal{M}} z$ and $z(\beta)^{\mathcal{M}} y\}$
  - $(\alpha \cup \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \cup (\beta)^{\mathcal{M}}$
  - $(\alpha^\star)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } n \in \mathbb{N}$ and there exists $z_0, \ldots, z_n \in W$ such that $x = z_0(\alpha)^{\mathcal{M}} \ldots (\alpha)^{\mathcal{M}} z_n = y\}$
  - $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)\colon \text{there exists } z, t, u, v \in W$ such that $x \star (z, t)$, $y \star (u, v)$, $z(\alpha)^{\mathcal{M}} u$ and $t(\beta)^{\mathcal{M}} v\}$

Separation and parallel composition

PRSPDL

Truth conditions according to Frias (2002)

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(a)^{\mathcal{M}} = R(a)$
  - $(\phi?)^{\mathcal{M}} = \{(x, y)\colon x = y$ and $y \in (\phi)^{\mathcal{M}}\}$
  - $(s_1)^{\mathcal{M}} = \{(x, y)\colon$ there exists $z \in W$ such that $y \star (x, z)\}$
  - $(s_2)^{\mathcal{M}} = \{(x, y)\colon$ there exists $z \in W$ such that $y \star (z, x)\}$
  - $(r_1)^{\mathcal{M}} = \{(x, y)\colon$ there exists $z \in W$ such that $x \star (y, z)\}$
  - $(r_2)^{\mathcal{M}} = \{(x, y)\colon$ there exists $z \in W$ such that $x \star (z, y)\}$
  - $(\alpha; \beta)^{\mathcal{M}} = \{(x, y)\colon$ there exists $z \in W$ such that $x(\alpha)^{\mathcal{M}}z$ and $z(\beta)^{\mathcal{M}}y\}$
  - $(\alpha \cup \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \cup (\beta)^{\mathcal{M}}$
  - $(\alpha^{\star})^{\mathcal{M}} = \{(x, y)\colon$ there exists $n \in \mathbb{N}$ and there exists $z_0, \dots, z_n \in W$ such that $x = z_0(\alpha)^{\mathcal{M}} \dots (\alpha)^{\mathcal{M}} z_n = y\}$
  - $(\alpha \parallel \beta)^{\mathcal{M}} = \{(x, y)\colon$ there exists $z, t \in W$ such that $y \star (z, t)$, $x(\alpha)^{\mathcal{M}}z$ and $x(\beta)^{\mathcal{M}}t\}$

Separation and parallel composition

PRSPDL

Truth conditions

- in a model $\mathcal{M} = (W, R, *, V)$ we define
  - $(p)^{\mathcal{M}} = V(p)$
  - $(\bot)^{\mathcal{M}}$ is empty
  - $(\neg\phi)^{\mathcal{M}} = W \setminus (\phi)^{\mathcal{M}}$
  - $(\phi \vee \psi)^{\mathcal{M}} = (\phi)^{\mathcal{M}} \cup (\psi)^{\mathcal{M}}$
  - $([\alpha]\phi)^{\mathcal{M}} = \{x\text{: for all } y \in W, \text{ if } x(\alpha)^{\mathcal{M}}y, y \in (\alpha)^{\mathcal{M}}\}$

Separation and parallel composition

PRSPDL

A model $\mathcal{M} = (W, R, *, V)$ is said to be separated iff

- if $x * (y, z)$ and $x * (t, u)$, $y = t$ and $z = u$

A model $\mathcal{M} = (W, R, *, V)$ is said to be deterministic iff

- if $x * (z, t)$ and $y * (z, t)$, $x = y$

In a separated model $\mathcal{M} = (W, R, *, V)$ we have

- if $x(s_1)^{\mathcal{M}}z$ and $z(r_1)^{\mathcal{M}}y$, $x = y$
- if $x(s_2)^{\mathcal{M}}z$ and $z(r_2)^{\mathcal{M}}y$, $x = y$

In a deterministic separated model $\mathcal{M} = (W, R, *, V)$ we have

- if $x(r_1)^{\mathcal{M}}z$, $z(s_1)^{\mathcal{M}}y$, $x(r_2)^{\mathcal{M}}t$ and $t(s_2)^{\mathcal{M}}y$, $x = y$

Separation and parallel composition

PRSPDL

Restriction of the syntax

- $\alpha, \beta ::= a \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

Bibliography

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Separation and parallel composition

PRSPDL

Axiomatization

- all tautologies   modus ponens   necessitation
- $[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$
- $\langle r_1\rangle\phi \to [r_1]\phi \qquad \langle r_2\rangle\phi \to [r_2]\phi$
- $\phi \to [s_1]\langle r_1\rangle\phi \quad \phi \to [s_2]\langle r_2\rangle\phi \quad \phi \to [r_1]\langle s_1\rangle\phi \quad \phi \to [r_2]\langle s_2\rangle\phi$
- $\langle s_1\rangle\top \leftrightarrow \langle s_2\rangle\top \qquad \langle r_1\rangle\top \leftrightarrow \langle r_2\rangle\top$
- $\langle s_1; r_1\rangle\phi \to [s_1; r_1]\phi \qquad \langle s_2; r_2\rangle\phi \to [s_2; r_2]\phi$
- $[s_1; r_2]\phi \to \phi$
- $\phi \to [s_1; r_2]\langle s_1; r_2\rangle\phi$
- $[s_1; r_2]\phi \to [s_1; r_2][s_1; r_2]\phi$
- $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$
- $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$

Separation and parallel composition

PRSPDL

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

For all $i \in \{1, 2\}$ and for all s_i-free programs α

- the programs s_i and α are not equally interpreted in all separated models

For all $i \in \{1, 2\}$ and for all r_i-free programs α

- the programs r_i and α are not equally interpreted in all separated models

Separation and parallel composition

PRSPDL

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha; \beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

For all atomic programs a, b and for all $\parallel$-free programs α

- the programs $a \parallel b$ and α are not equally interpreted in all separated models

Separation and parallel composition

PRSPDL

Syntax

- $\alpha, \beta ::= a \mid \phi? \mid s_1 \mid s_2 \mid r_1 \mid r_2 \mid (\alpha;\beta) \mid (\alpha \cup \beta) \mid \alpha^\star \mid (\alpha \parallel \beta)$
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi$

The following expressions are equally interpreted in all separated models for each programs α, β, for each formulas ϕ and for each atomic formulas p not occurring in α, β, ϕ

- $\langle \alpha \parallel \beta \rangle \phi$
- $\forall p \, (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))$

Separation and parallel composition

PRSPDL

The following expressions are equally interpreted in all separated models for each programs α, β, for each formulas ϕ and for each atomic formulas p not occurring in α, β, ϕ

- $\langle \alpha \parallel \beta \rangle \phi$
- $\forall p \, (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))$

Axiom

- $\langle \alpha \parallel \beta \rangle \phi \rightarrow (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge \psi) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg \psi))$

Inference rule

- from $\chi \rightarrow (\langle r_1 \rangle \langle \alpha \rangle \langle s_1 \rangle (\phi \wedge p) \vee \langle r_2 \rangle \langle \beta \rangle \langle s_2 \rangle (\phi \wedge \neg p))$, infer $\chi \rightarrow \langle \alpha \parallel \beta \rangle \phi$

Open problems

Truth conditions of Benevides *et al.* (2011)

- Decidability/complexity of satisfiability for the restriction considered by Benevides *et al.* (2011)
- Decidability/complexity of satisfiability for the full language
- Tableau calculus for the restriction considered by Benevides *et al.* (2011)
- Tableau calculus for the full language
- Axiomatization of validity for the full language

Truth conditions of Frias (2002)

- Same issues

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