# Quelques pas autour de la logique spatiale d’arbres

# Contents

# Motivation

Semi-structured data

- are unstructured data
- are usually modeled in terms of labeled trees
- are amenable to be accessed through query languages

# Motivation

Ambient logics

- are modal logics
- are used to describe the properties of mobile computations
- are based on a rich collection of spatio-temporal operators

# Motivation

The structural part of ambient logics

- is a logic of trees
- is a logic designed to describe properties of labeled trees
- is particularly suitable to describe semi-structured data

# Motivation

Example

- $ARTICLES = article[author[Biri] \mid author[Galmiche] \mid title[Models] \mid journal[JLC]] \mid article[author[Galmiche] \mid author[LarcheyWendling] \mid title[Expressivity] \mid conference[FSTTCS\ 2006]]$
- $from\ ARTICLES \models \cdot article[X],\ X \models \cdot author[Galmiche]$ $select\ paper[X]$

Reference


- Cardelli, L., Ghelli, G.: A query language based on the ambient logic. ESOP 2001. Springer (2001) 1–22.

# Information trees

Information trees

- are nested multisets
- correspond to unordered trees

Given a set $\Lambda$ of labels, $\mathcal{IT}$ is the least collection such that

- $\emptyset$ is in $\mathcal{IT}$
- if $m$ is a label in $\Lambda$ and $M$ is in $\mathcal{IT}$, then $\{\langle m, M\rangle\}$ is in $\mathcal{IT}$
- if $M$ and $N$ are in $\mathcal{IT}$, then $M \uplus N$ is in $\mathcal{IT}$

# Information trees

Information terms

- are terms borrowed from the ambient calculus
- denote information trees

Given a set $\Lambda$ of labels, the set of information terms is the least collection such that

- $0$ is an information term
- if $m$ is a label in $\Lambda$ and $F$ is an information term, then $m[F]$ is an information term
- if $F$ and $G$ are information terms, then $(F \mid G)$ is an information term

# Information trees

Information terms and their information tree meaning

- $[\![0]\!] = \emptyset$
- $[\![m[F]]\!] = \{\langle m, [\![F]\!]\rangle\}$
- $[\![F \mid G]\!] = [\![F]\!] \uplus [\![G]\!]$

# Information trees

Congruence over information terms

- $F \equiv F$
- if $F \equiv G$, then $G \equiv F$
- if $F \equiv G$ and $G \equiv H$, then $F \equiv H$
- if $F \equiv G$, then $m[F] \equiv m[G]$
- if $F \equiv G$, then $F \mid H \equiv G \mid H$
- if $F \equiv G$, then $H \mid F \equiv H \mid G$
- $F \mid 0 \equiv F$
- $0 \mid F \equiv F$
- $F \mid G \equiv G \mid F$
- $(F \mid G) \mid H \equiv F \mid (G \mid H)$

# The tree logic

Expressions

- $\alpha ::= m \mid x$

where $m$ is a label in $\Lambda$ and $x$ is a label variable

Formulas

- $\phi ::= \xi \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid 0 \mid \alpha[\phi] \mid (\phi \mid \psi) \mid \mathcal{X} \mid \exists x \cdot \phi \mid \exists \mathcal{X} \cdot \phi \mid \alpha \sim \beta \mid \mu\xi \cdot \phi$

where $\xi$ is a recursion variable and $\mathcal{X}$ is a tree variable

Reference

- Cardelli, L., Gordon, A.: Anytime, anywhere: modal logics for mobile ambients. POPL 2000. ACM (2000) 365–377.

# The tree logic

Semantic map

- interpretation $\rho : x \mapsto \rho(x) \in \Lambda$ of label variables
- interpretation $\sigma : \mathcal{X} \mapsto \sigma(\mathcal{X}) \in \mathcal{IT}$ of tree variables
- interpretation $\tau : \xi \mapsto \tau(\xi) \in 2^{\mathcal{IT}}$ of recursion variables

Interpretation of formulas

- $[\![ \cdot ]\!]_{\rho,\sigma,\tau} : \phi \mapsto [\![\phi]\!]_{\rho,\sigma,\tau} \in 2^{\mathcal{IT}}$

Satisfaction

- $F$ satisfies $\phi$ under $\rho$, $\sigma$ and $\tau$, denoted $F \models_{\rho,\sigma,\tau} \phi$, iff $[\![F]\!] \in [\![\phi]\!]_{\rho,\sigma,\tau}$

# The tree logic

Formulas as sets of information trees

- $[\![\xi]\!]_{\rho,\sigma,\tau} = \tau(\xi)$
- $[\![\bot]\!]_{\rho,\sigma,\tau} = \emptyset$
- $[\![\neg\phi]\!]_{\rho,\sigma,\tau} = \mathcal{IT} \setminus [\![\phi]\!]_{\rho,\sigma,\tau}$
- $[\![\phi \vee \psi]\!]_{\rho,\sigma,\tau} = [\![\phi]\!]_{\rho,\sigma,\tau} \cup [\![\psi]\!]_{\rho,\sigma,\tau}$
- $[\![0]\!]_{\rho,\sigma,\tau} = \{\emptyset\}$
- $[\![\alpha[\phi]]\!]_{\rho,\sigma,\tau} = \{\langle\rho(\alpha), M\rangle\colon M \text{ is in } [\![\phi]\!]_{\rho,\sigma,\tau}\}$
- $[\![\phi \mid \psi]\!]_{\rho,\sigma,\tau} = \{M \uplus N\colon M \text{ is in } [\![\phi]\!]_{\rho,\sigma,\tau} \text{ and } N \text{ is in } [\![\psi]\!]_{\rho,\sigma,\tau}\}$
- $[\![\mathcal{X}]\!]_{\rho,\sigma,\tau} = \{\sigma(\mathcal{X})\}$
- $[\![\exists x \cdot \phi]\!]_{\rho,\sigma,\tau} = \bigcup\{[\![\phi]\!]_{\rho[x:=m],\sigma,\tau}\colon m \text{ is a label in } \Lambda\}$
- $[\![\exists \mathcal{X} \cdot \phi]\!]_{\rho,\sigma,\tau} = \bigcup\{[\![\phi]\!]_{\rho,\sigma[\mathcal{X}:=M],\tau}\colon M \text{ is in } \mathcal{IT}\}$
- $[\![\alpha \sim \beta]\!]_{\rho,\sigma,\tau} = \text{if } \rho(\alpha) = \rho(\beta) \text{ then } \mathcal{IT} \text{ else } \emptyset$
- $[\![\mu\xi \cdot \phi]\!]_{\rho,\sigma,\tau} = \bigcap\{S\colon S \text{ contains } [\![\phi]\!]_{\rho,\sigma,\tau[\xi:=S]}\}$

# The tree logic

Some properties of satisfaction

- $F \models_{\rho,\sigma,\tau} \xi$ iff $[\![F]\!]$ is in $\tau(\xi)$
- $F \not\models_{\rho,\sigma,\tau} \bot$
- $F \models_{\rho,\sigma,\tau} \neg\phi$ iff $F \not\models_{\rho,\sigma,\tau} \phi$
- $F \models_{\rho,\sigma,\tau} \phi \vee \psi$ iff $F \models_{\rho,\sigma,\tau} \phi$ or $F \models_{\rho,\sigma,\tau} \psi$
- $F \models_{\rho,\sigma,\tau} 0$ iff $F \equiv 0$
- $F \models_{\rho,\sigma,\tau} \alpha[\phi]$ iff for some $G$, $F \equiv \rho(\alpha)[G]$ and $G \models_{\rho,\sigma,\tau} \phi$
- $F \models_{\rho,\sigma,\tau} \phi \mid \psi$ iff for some $G, H$, $F \equiv G \mid H$, $G \models_{\rho,\sigma,\tau} \phi$ and $H \models_{\rho,\sigma,\tau} \psi$
- $F \models_{\rho,\sigma,\tau} \mathcal{X}$ iff $[\![F]\!]$ equals $\sigma(\mathcal{X})$
- $F \models_{\rho,\sigma,\tau} \exists x \cdot \phi$ iff for some label $m$ in $\Lambda$, $F \models_{\rho[x:=m],\sigma,\tau} \phi$
- $F \models_{\rho,\sigma,\tau} \exists \mathcal{X} \cdot \phi$ iff for some $M$ in $\mathcal{IT}$, $F \models_{\rho,\sigma[\mathcal{X}:=M],\tau} \phi$
- $F \models_{\rho,\sigma,\tau} \alpha \sim \beta$ iff $\rho(\alpha)$ equals $\rho(\beta)$
- $F \models_{\rho,\sigma,\tau} \mu\xi \cdot \phi$ iff $F \models_{\rho,\sigma,\tau} \phi[\xi/\mu\xi \cdot \phi]$

# The tree logic

Derived formulas

- $\alpha[\Rightarrow \phi]$ is $\neg\alpha[\neg\phi]$
- $(\phi \parallel \psi)$ is $\neg(\neg\phi \mid \neg\psi)$
- $\forall x \cdot \phi$ is $\neg\exists x \cdot \neg\phi$
- $\forall \mathcal{X} \cdot \phi$ is $\neg\exists \mathcal{X} \cdot \neg\phi$
- $\nu\xi \cdot \phi$ is $\neg\mu\xi \cdot \neg\phi[\xi/\neg\xi]$
- $\phi^\star$ is $\mu\xi \cdot 0 \vee (\phi \mid \xi)$

# The tree logic

Some valid equivalences

- $\alpha[\phi] \leftrightarrow \alpha[\top] \wedge \alpha[\Rightarrow \phi]$
- $\alpha[\Rightarrow \phi] \leftrightarrow (\alpha[\top] \rightarrow \alpha[\phi])$
- $\alpha[\bot] \leftrightarrow \bot$
- $\alpha[\Rightarrow \top] \leftrightarrow \top$
- $\alpha[\phi \vee \psi] \leftrightarrow \alpha[\phi] \vee \alpha[\psi]$
- $\alpha[\Rightarrow \phi \vee \psi] \leftrightarrow \alpha[\Rightarrow \phi] \vee \alpha[\Rightarrow \psi]$
- $\alpha[\phi \wedge \psi] \leftrightarrow \alpha[\phi] \wedge \alpha[\psi]$
- $\alpha[\Rightarrow \phi \wedge \psi] \leftrightarrow \alpha[\Rightarrow \phi] \wedge \alpha[\Rightarrow \psi]$
- $\alpha[Qx \cdot \phi] \leftrightarrow Qx \cdot \alpha[\phi]$ if $\alpha \neq x$
- $\alpha[\Rightarrow Qx \cdot \phi] \leftrightarrow Qx \cdot \alpha[\Rightarrow \phi]$ if $\alpha \neq x$

# The tree logic

Some valid equivalences

- $\phi \mid \bot \leftrightarrow \bot$
- $\phi \parallel \top \leftrightarrow \top$
- $\phi \mid (\psi \vee \chi) \leftrightarrow \phi \mid \psi \vee \phi \mid \chi$
- $\phi \parallel (\psi \wedge \chi) \leftrightarrow \phi \parallel \psi \wedge \phi \parallel \chi$
- $\phi \mid \psi \leftrightarrow \psi \mid \phi$
- $\phi \parallel \psi \leftrightarrow \psi \parallel \phi$
- $(\phi \mid \psi) \mid \chi \leftrightarrow \phi \mid (\psi \mid \chi)$
- $(\phi \parallel \psi) \parallel \chi \leftrightarrow \phi \parallel (\psi \parallel \chi)$
- $\phi \mid Qx \cdot \psi \leftrightarrow Qx \cdot \phi \mid \psi$ if $x \notin FV(\phi)$
- $\phi \parallel Qx \cdot \psi \leftrightarrow Qx \cdot \phi \parallel \psi$ if $x \notin FV(\phi)$

# The tree logic

Questions

- axiomatization/completess ?
- decidability/complexity ?
- variants
  - $\phi ::= \xi \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid 0 \mid \alpha[\phi] \mid (\phi \mid \psi) \mid \mu\xi \cdot \phi$
  - $\phi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid 0 \mid \alpha[\phi] \mid (\phi \mid \psi) \mid \phi^\star$

# Adjuncts

Formulas

- $\phi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid 0 \mid m[\phi] \mid (\phi \mid \psi) \mid \phi@m \mid (\phi \triangleright \psi)$

Reference


- Calcagno, C., Cardelli, L., Gordon, A.: Deciding validity in a spatial logic for trees. Journal of Functional Programming **15** (2005) 543–572.
- Lozes, E.: Elimination of spatial connectives in static spatial logics. Theoretical Computer Science **330** (2005) 475–499.

# Adjuncts

Interpretation of formulas

- $[\![ \cdot ]\!] : \phi \mapsto [\![\phi]\!] \in 2^{\mathcal{IT}}$

Satisfaction

- $F$ satisfies $\phi$, denoted $F \models \phi$, iff $[\![F]\!] \in [\![\phi]\!]$

# Adjuncts

Formulas as sets of information trees

- $[\![\bot]\!] = \emptyset$
- $[\![\neg\phi]\!] = \mathcal{IT} \setminus [\![\phi]\!]$
- $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$
- $[\![0]\!] = \{\emptyset\}$
- $[\![m[\phi]]\!] = \{\langle m, M\rangle\text{: } M \text{ is in } [\![\phi]\!]\}$
- $[\![\phi \mid \psi]\!] = \{M \uplus N\text{: } M \text{ is in } [\![\phi]\!] \text{ and } N \text{ is in } [\![\psi]\!]\}$
- $[\![\phi @ m]\!] = \{M\text{: } \langle m, M\rangle \text{ is in } [\![\phi]\!]\}$
- $[\![\phi \triangleright \psi]\!] = \{M\text{: for every } N \text{ in } [\![\phi]\!],\ M \uplus N \text{ is in } [\![\psi]\!]\}$

# Adjuncts

Information trees as formulas

- $\varphi(0) = 0$
- $\varphi(m[F]) = m[\varphi(F)]$
- $\varphi(F \mid G) = \varphi(F) \mid \varphi(G)$

Validity vs model checking

- $\phi$ is valid iff $0 \models \top \triangleright \phi$
- $F \models \phi$ iff $\varphi(F) \rightarrow \phi$ is valid

# Adjuncts

Bisimilarity between information terms

- if $F \simeq_i G$ then either $i = 0$, or
  - $F \equiv 0$ iff $G \equiv 0$
  - for every label $m$ in $\Lambda$ and for every information term $F'$, if $F \equiv m[F']$ then for some information term $G'$, $G \equiv m[G']$ and $F' \simeq_{i-1} G'$
  - for every label $m$ in $\Lambda$ and for every information term $G'$, if $G \equiv m[G']$ then for some information term $F'$, $F \equiv m[F']$ and $F' \simeq_{i-1} G'$
  - for every information term $F', F''$, if $F \equiv F' \mid F''$ then for some information term $G', G''$, $G \equiv G' \mid G''$, $F' \simeq_{i-1} G'$ and $F'' \simeq_{i-1} G''$
  - for every information term $G', G''$, if $G \equiv G' \mid G''$ then for some information term $F', F''$, $F \equiv F' \mid F''$, $F' \simeq_{i-1} G'$ and $F'' \simeq_{i-1} G''$

# Adjuncts

Ultrametric distance between information terms

- $d(F,G) = \sup\{2^{-i} \colon i \in \mathbb{N} \text{ is such that } F \not\simeq_i G\}$

Remark

- $d(F,G) < 2^{-i}$ iff $F \simeq_i G$

Properties of the ultrametric distance

- complete
- totally bounded
- compact
- separable

# Adjuncts

Theorem (Calcagno *et al.*, 2005; Lozes, 2005)

- every formula is equivalent to some adjunct-free formula
- validity and model checking are decidable

# Adjuncts

Questions

- axiomatization/completess ?
- decidability/complexity ?
- variants
  - $\phi ::= \xi \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid 0 \mid \alpha[\phi] \mid (\phi \mid \psi) \mid \phi @ m \mid (\phi \triangleright \psi) \mid \mu\xi \cdot \phi$
  - $\phi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid 0 \mid \alpha[\phi] \mid (\phi \mid \psi) \mid \phi @ m \mid (\phi \triangleright \psi) \mid \phi^\star$

# Knowledge

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- zzzz

# Notes

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