A Connection-based Characterization of Bi-Intuitionistic Validity

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CADE 23, Wroclaw, 2011

Introduction

Context & Motivations

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- Bi-Intuitionistic Logic (BiInt)
  - conservative extension of IL
  - duality: implication $\rightarrow$ / co-implication $\prec$

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- Recent works
  - Goré & Postniece (2008): cut-elimination via nested sequents
  - Pinto & Uustalu (2009): cut-elimination via labelled sequents

Our Contributions

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  - R-Graphs: unbiased graphical structure
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  - free variables
  - variable splitting
- Variable binding algorithm
  - solving R-Graphs admissibility constraints
  - alternative to IL prefix-unification

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- Sequent calculus (Dragalin style)

multi-conclusioned LJ + rules for $\prec$

Multi-Conclusioned Sequent Calculus

$$\frac{}{\Gamma, \bot \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot_R$$

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{ax} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ \text{cut}$$

$$\frac{\Gamma, A \to B \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ \to_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B, \Delta}\ \to_R$$

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$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\ \land_L \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta}\ \land_R$$

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Cut Elimination Failure

$$\mathrm{P} \vdash \mathrm{R} \rightarrow ((\mathrm{P} \prec \mathrm{Q}) \wedge \mathrm{R}), \mathrm{Q}$$

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$$P \vdash R \to ((P \prec Q) \land R), Q$$

- Without cut

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- Kripke model $\mathcal{M} = \langle M, \sqsubseteq, [\![\cdot]\!]\rangle$: **POSET of worlds**

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  - $m \models \bot$ never
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  - $m \models \mathrm{A} \vee \mathrm{B}$ iff $m \models \mathrm{A}$ or $m \models \mathrm{B}$
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- Validity: $\models \mathrm{A}$ iff $\mathcal{M} \models \mathrm{A}$ for all Kripke models $\mathcal{M}$

Indexing Formulas

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- Principal and secondary types

α	α_1	α_2	β	β_1	β_2
$(A \wedge B)^+$	A^+	B^+	$(A \wedge B)^-$	A^-	B^-
$(A \vee B)^-$	A^-	B^-	$(A \vee B)^+$	A^+	B^+
$(A \rightarrow B)^-$	A^+	B^-	$(A \rightarrow B)^+$	A^-	B^+
$(A \prec B)^+$	A^+	B^-	$(A \prec B)^-$	A^-	B^+

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$(\mathrm{A} \to \mathrm{B})^{-}$	A^{+}	B^{-}	$(\mathrm{A} \to \mathrm{B})^{+}$	A^{-}	B^{+}
$(\mathrm{A} \prec \mathrm{B})^{+}$	A^{+}	B^{-}	$(\mathrm{A} \prec \mathrm{B})^{-}$	A^{-}	B^{+}

- Special bi-intuitionistic types

	itype
$(\mathrm{A} \to \mathrm{B})^{+}$	ϕ
$(\mathrm{A} \to \mathrm{B})^{-}$	ψ

	itype
$(\mathrm{A} \prec \mathrm{B})^{+}$	$\overline{\psi}$
$(\mathrm{A} \prec \mathrm{B})^{-}$	$\overline{\phi}$

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- Depth first exploration of A^- $\begin{cases} \text{respecting the syntactic structure} \\ \psi\text{- and } \overline{\psi}\text{-formulas: } \textbf{constants} \\ \phi\text{- and } \overline{\phi}\text{-formulas: } \textbf{variables} \\ \text{everything else: natural numbers} \end{cases}$

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- Example: indexing $\mathrm{C} = (((\mathrm{R} \vee \mathrm{P}) \prec \mathrm{Q}) \wedge \neg \mathrm{P}) \rightarrow ((\mathrm{P} \rightarrow \mathrm{R}) \prec \mathrm{Q})$

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$$(((\mathrm{R}_2^+ \vee_1^+ \mathrm{P}_3^+) \prec_{\overline{a}}^+ \mathrm{Q}_4^-) \wedge_0^+ \neg_x^- \mathrm{P}_5^-) \rightarrow_a^- ((\mathrm{P}_6^+ \rightarrow_b^- \mathrm{R}_7^-) \prec_{\overline{x}}^- \mathrm{Q}_8^+)$$

Formula Tree

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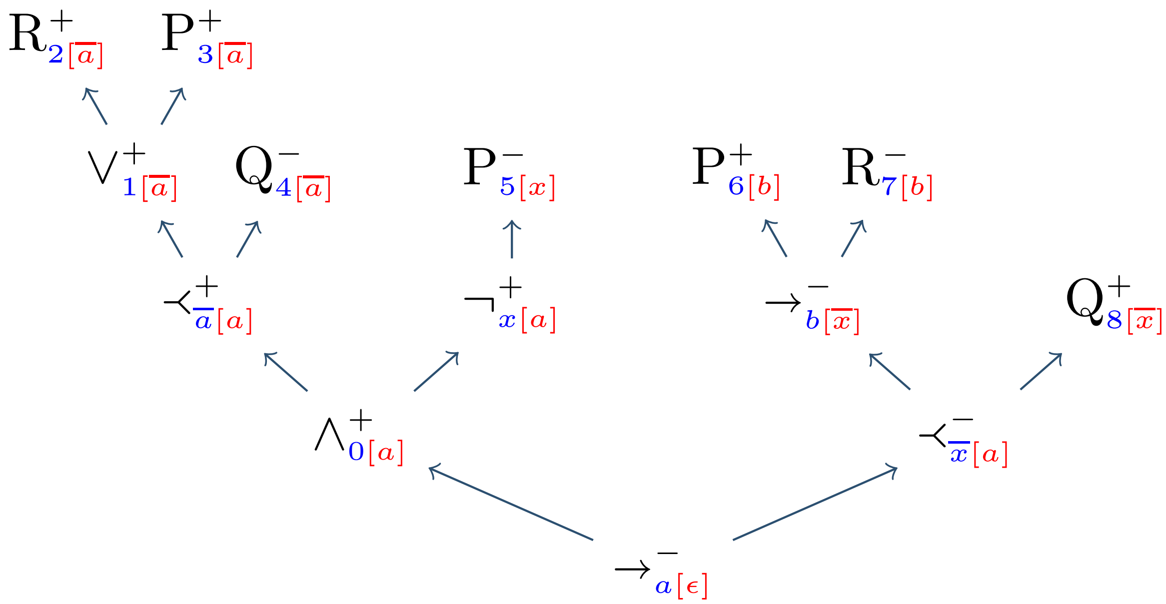

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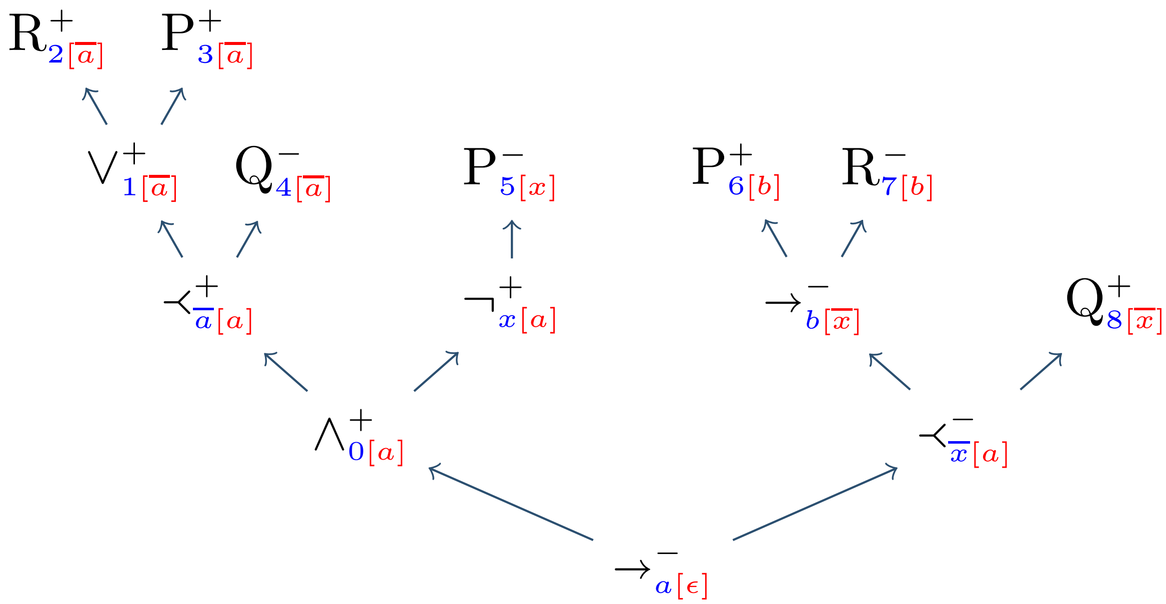

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- Strict **partial ordering** on indexes (subformula ordering)

$i \ll j$ iff $j \in i$ iff i dominates j in the syntax tree

Multiplicity

- Sequent calculus: implicit **contraction** rules

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- **Multiplicity** $\mu()$: ϕ- and $\overline{\phi}$-formulas $\to \mathbb{N}$ (**number of copies**)

$$\mu(A) = n \Longrightarrow A \wedge \underbrace{A \wedge \ldots \wedge A}_{n \text{ times}}$$

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- Sequent calculus: implicit **contraction** rules

$$\frac{\Gamma, A \to B \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \to_L$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash A \prec B, \Delta}{\Gamma \vdash A \prec B, \Delta} \prec_R$$

- **Multiplicity** $\mu()$: ϕ- and $\overline{\phi}$-formulas $\to \mathbb{N}$ (**number of copies**)

$$\mu(A) = n \Longrightarrow A \wedge \underbrace{A \wedge \ldots \wedge A}_{n \text{ times}}$$

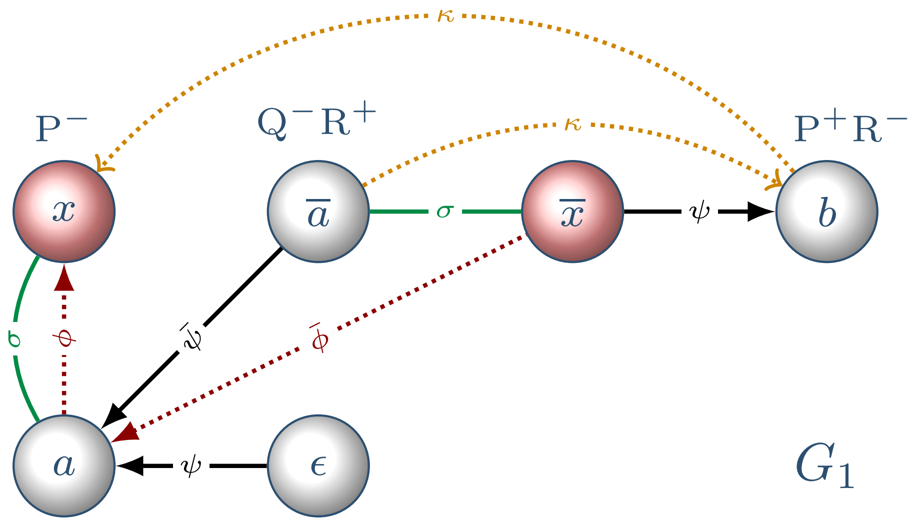
- Let $C = (P \to Q) \to (R \prec S)$, $\mu(P \to Q) = 1$ and $\mu(R \prec S) = 2$

$$\mu(C) = ((P \to Q) \wedge (P \to Q)) \to ((R \prec S) \wedge (R \prec S) \wedge (R \prec S))$$

BiInt R-Graphs

R-Graphs

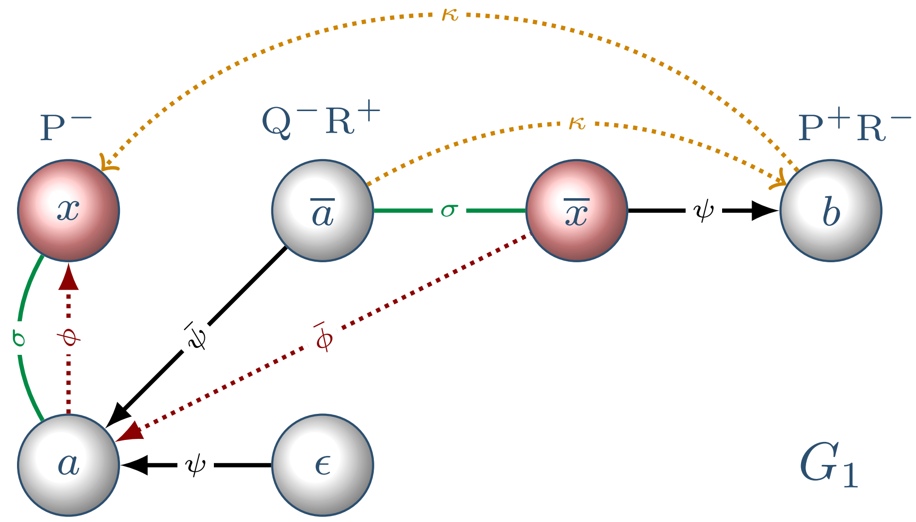
- R-Graph $G(V,E)$: directed graph such that
  - Vertices: named with labels (constants and variables)
  - V tagged with signed formulas
  - E tagged with letters $\mathcal{T} = \{\psi, \phi, \bar{\psi}, \bar{\phi}, \sigma, \kappa\}$



  - σ-edges: **bidirectional** between variable/constant vertices
  - κ-edges: **complementarity** link $\mathrm{A}^+/\mathrm{A}^-$ for some A

BiInt *R-Graphs*

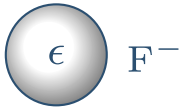
- BiInt R-Graphs (biRGs)
  - every ψ-edge has a ψ-vertex as its **target**
  - every $\bar{\psi}$-edge has a ψ-vertex as its **source**

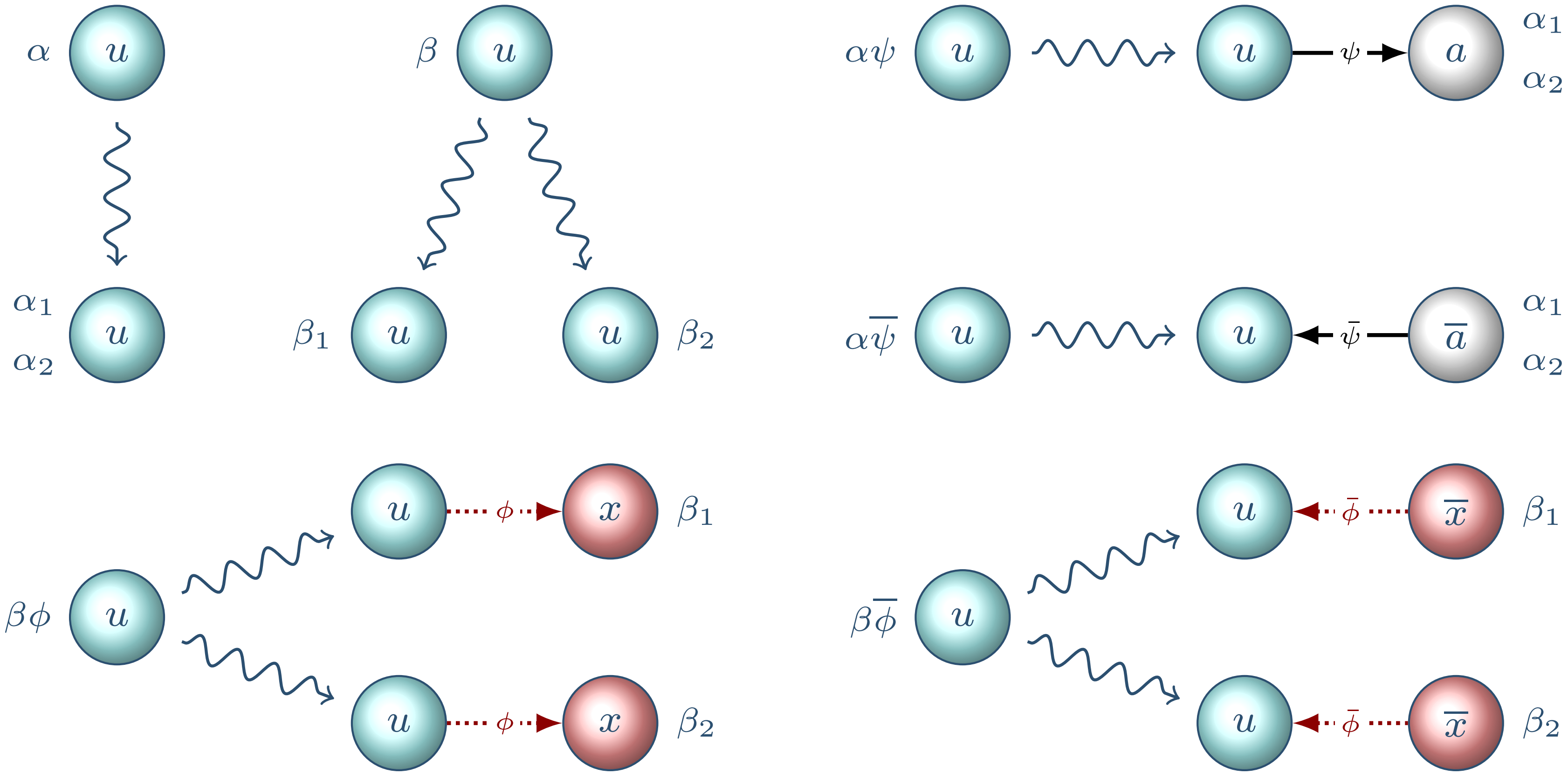


  - every ϕ-edge has a ϕ-vertex as its **target**
  - every $\bar{\phi}$-edge has a ϕ-vertex as its **source**

R-Graph Reductions

Reduction Rules (as R-Graphs)

Starting with the one node R-Graph  apply reduction rules



until stable under reduction (irreducibility): **collection of R-Graphs**

Reduction Rules (as Labelled Sequents)

Starting with $\vdash \mathrm{F}[\epsilon]$, apply reduction rules

$$\frac{}{\Gamma, \mathrm{A}[u] \vdash \mathrm{A}[u], \Delta}\ \mathrm{ax} \qquad \frac{}{\Gamma, \bot[u] \vdash \Delta}\ \bot_{\mathrm{L}}$$

$$\frac{\Gamma, \mathrm{A}[u], \mathrm{B}[u] \vdash \Delta}{\Gamma, (\mathrm{A} \wedge \mathrm{B})[u] \vdash \Delta}\ \wedge_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}[u], \Delta \qquad \Gamma \vdash \mathrm{B}[u], \Delta}{\Gamma \vdash (\mathrm{A} \wedge \mathrm{B})[u], \Delta}\ \wedge_{\mathrm{R}}$$

$$\frac{\Gamma, \mathrm{A}[u] \vdash \Delta \qquad \Gamma, \mathrm{B}[u] \vdash \Delta}{\Gamma, (\mathrm{A} \vee \mathrm{B})[u] \vdash \Delta}\ \vee_{\mathrm{L}} \qquad \frac{\Gamma \vdash \mathrm{A}[u], \mathrm{B}[u], \Delta}{\Gamma \vdash (\mathrm{A} \vee \mathrm{B})[u], \Delta}\ \vee_{\mathrm{R}}$$

$$\frac{\Gamma \vdash \mathrm{A}[x], \Delta \qquad \Gamma, \mathrm{B}[x] \vdash \Delta}{\Gamma, (\mathrm{A} \to \mathrm{B})[u] \vdash \Delta}\ u[\phi]x \qquad \frac{\Gamma, \mathrm{A}[a] \vdash \mathrm{B}[a], \Delta}{\Gamma \vdash (\mathrm{A} \to \mathrm{B})[u], \Delta}\ u[\psi]a$$

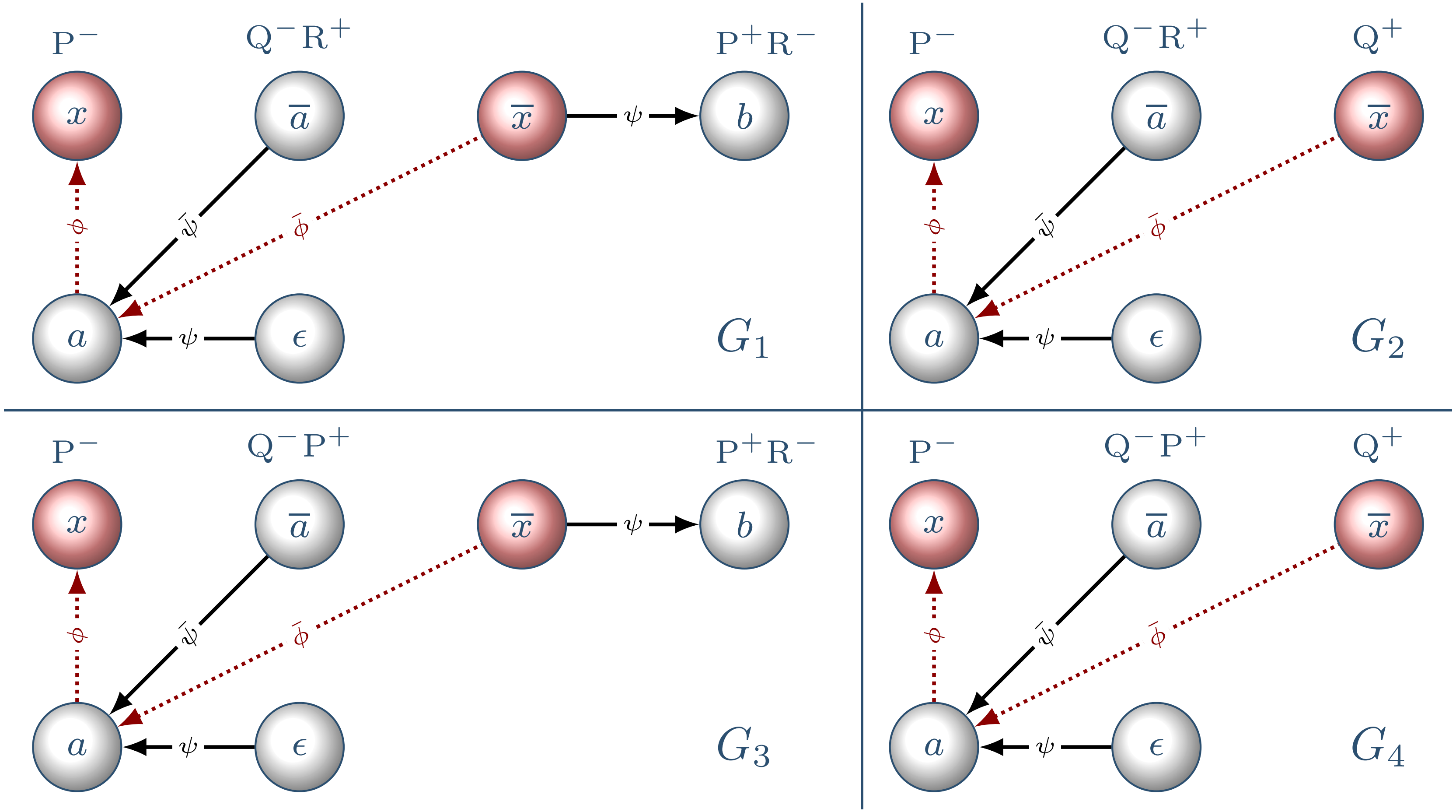
$$\frac{\Gamma \vdash \mathrm{A}[\overline{x}], \Delta \qquad \Gamma, \mathrm{B}[\overline{x}] \vdash \Delta}{\Gamma \vdash (\mathrm{A} \prec \mathrm{B})[u], \Delta}\ \overline{x}[\overline{\phi}]u \qquad \frac{\Gamma, \mathrm{A}[\overline{a}] \vdash \mathrm{B}[\overline{a}], \Delta}{\Gamma, (\mathrm{A} \prec \mathrm{B})[u] \vdash \Delta}\ \overline{a}[\overline{\psi}]u$$

until sequents contain only atomic formulas: **collection of initial sequents**

Reduction Example

Introduction
Bi-Intuitionistic Logic
Indexing Formulas
BiInt R-Graphs
R-Graph Reductions
❖ Reduction Rules (as R-Graphs)
❖ Reduction Rules (as Labelled Sequents)
❖ Reduction Example
Validity through R-Graphs
Counter-Model Extraction
Binding & Splitting Variables
Conclusion

$$(((\mathrm{R}^+_2 \vee^+_1 \mathrm{P}^+_3) \prec^+_{\overline{a}} \mathrm{Q}^-_4) \wedge^+_0 \neg^-_x \mathrm{P}^-_5) \to^-_a ((\mathrm{P}^+_6 \to^-_b \mathrm{R}^-_7) \prec^-_{\overline{x}} \mathrm{Q}^+_8)$$

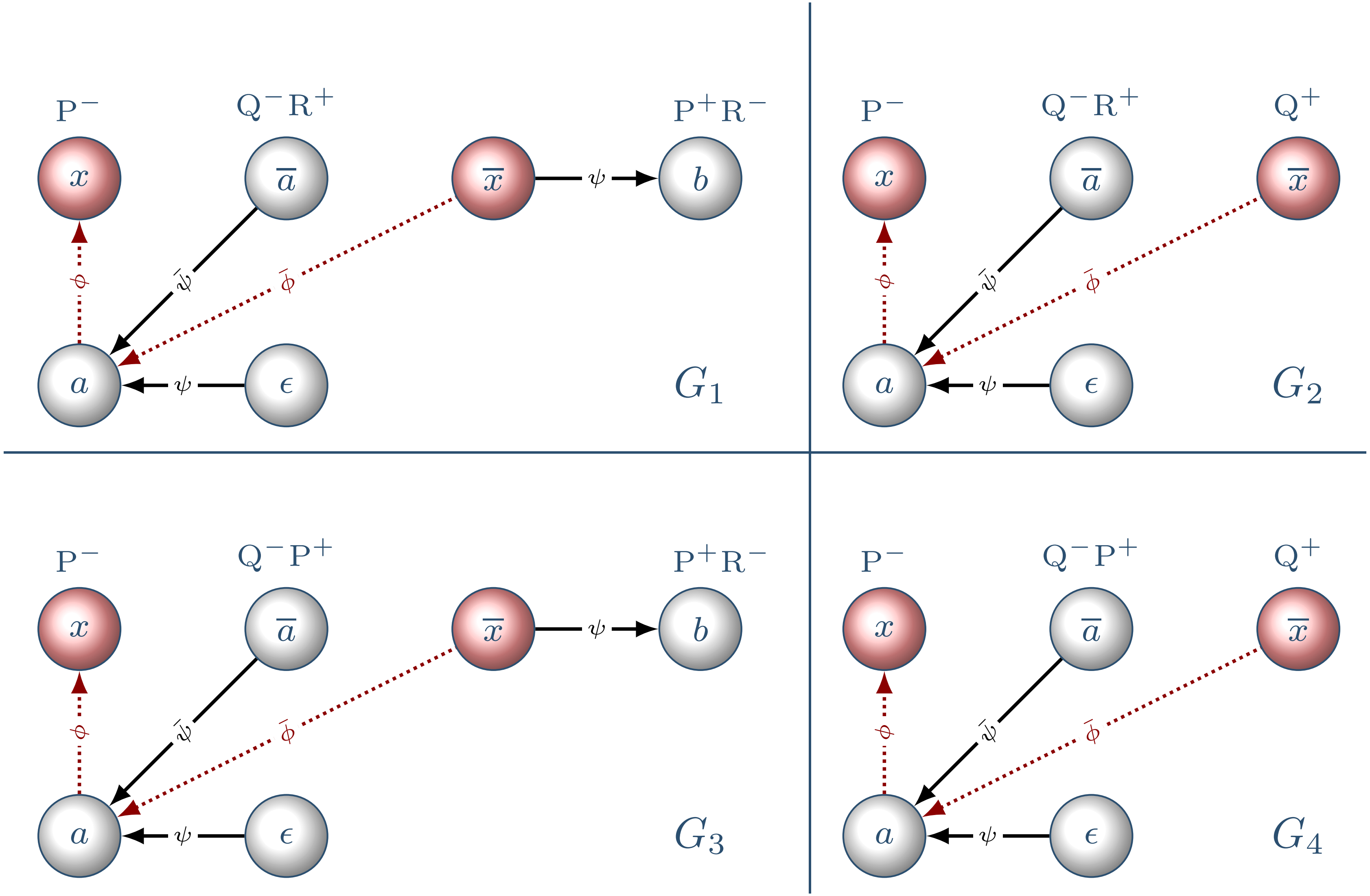


Reduction rules are **fully permutable**

Validity through R-Graphs

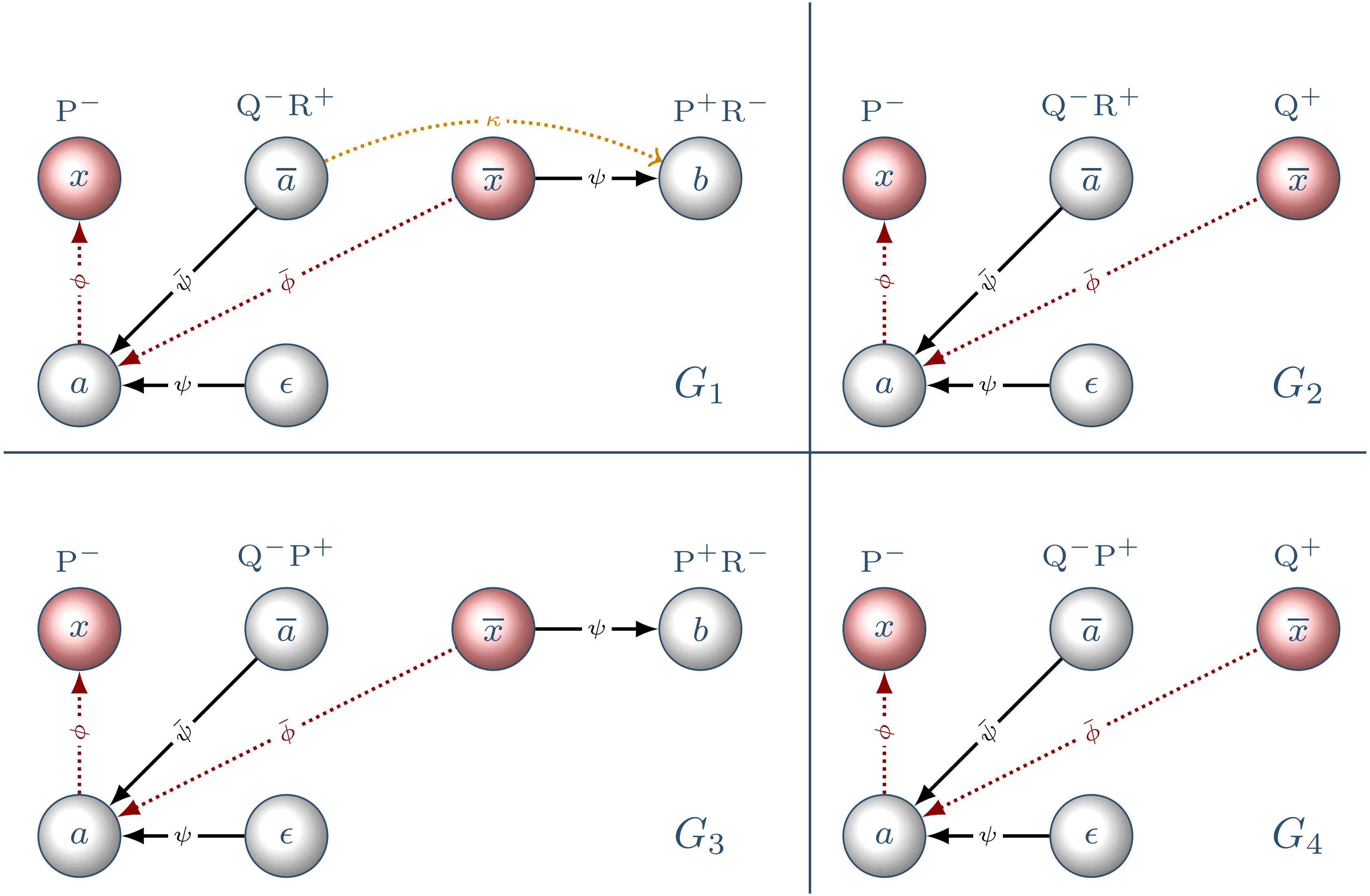
Binding Rules

$$(((\mathrm{R}^+_2 \vee^+_1 \mathrm{P}^+_3) \prec^+_{\bar{a}} \mathrm{Q}^-_4) \wedge^+_0 \neg^-_x \mathrm{P}^-_5) \to^-_a ((\mathrm{P}^+_6 \to^-_b \mathrm{R}^-_7) \prec^-_{\bar{x}} \mathrm{Q}^+_8)$$



Binding Rules

$$(((R^+_2 \vee^+_1 P^+_3) \prec^+_{\bar{a}} Q^-_4) \wedge^+_0 \neg^-_x P^-_5) \to^-_a ((P^+_6 \to^-_b R^-_7) \prec^-_{\bar{x}} Q^+_8)$$



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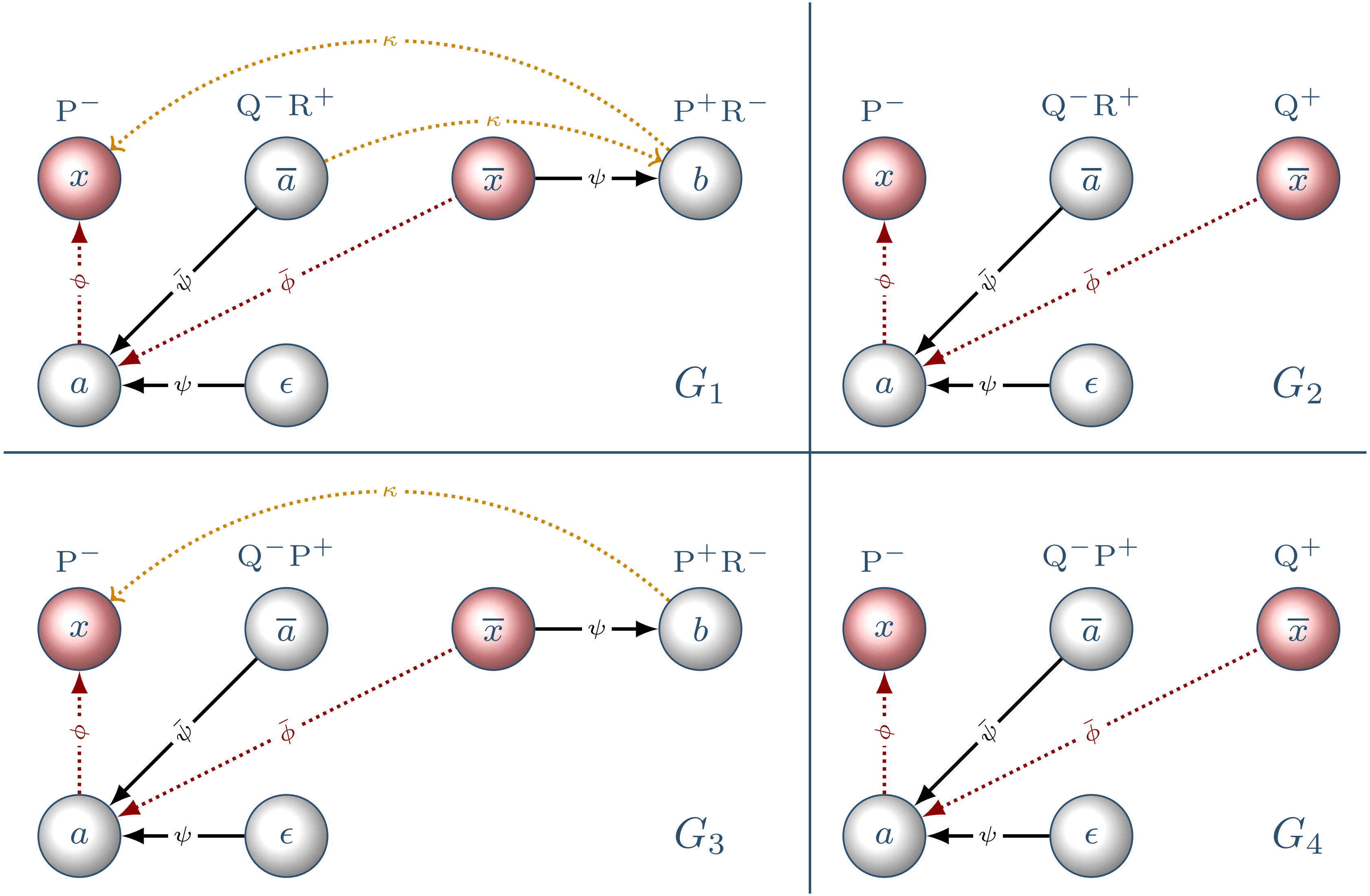

Binding Rules

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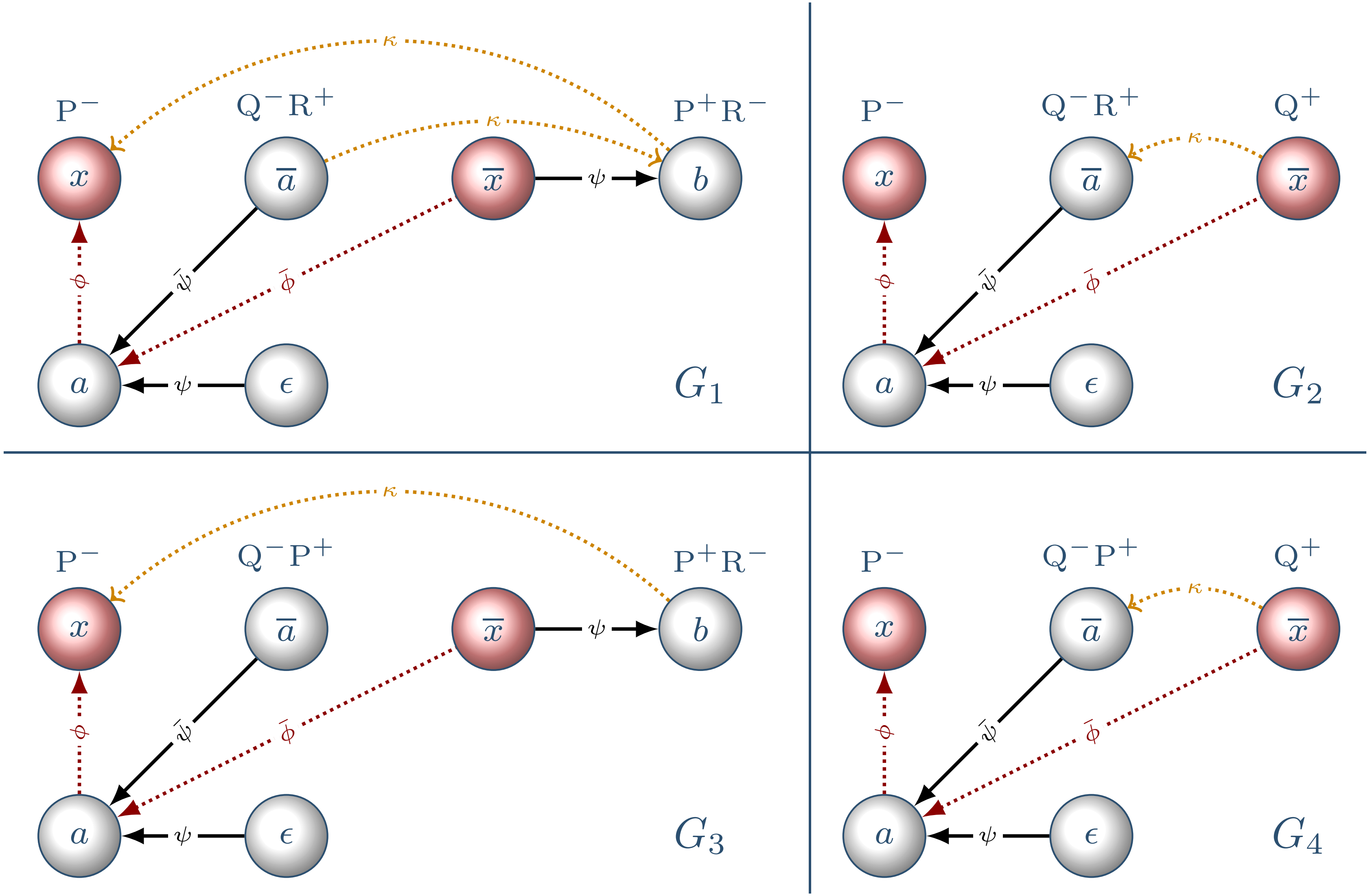

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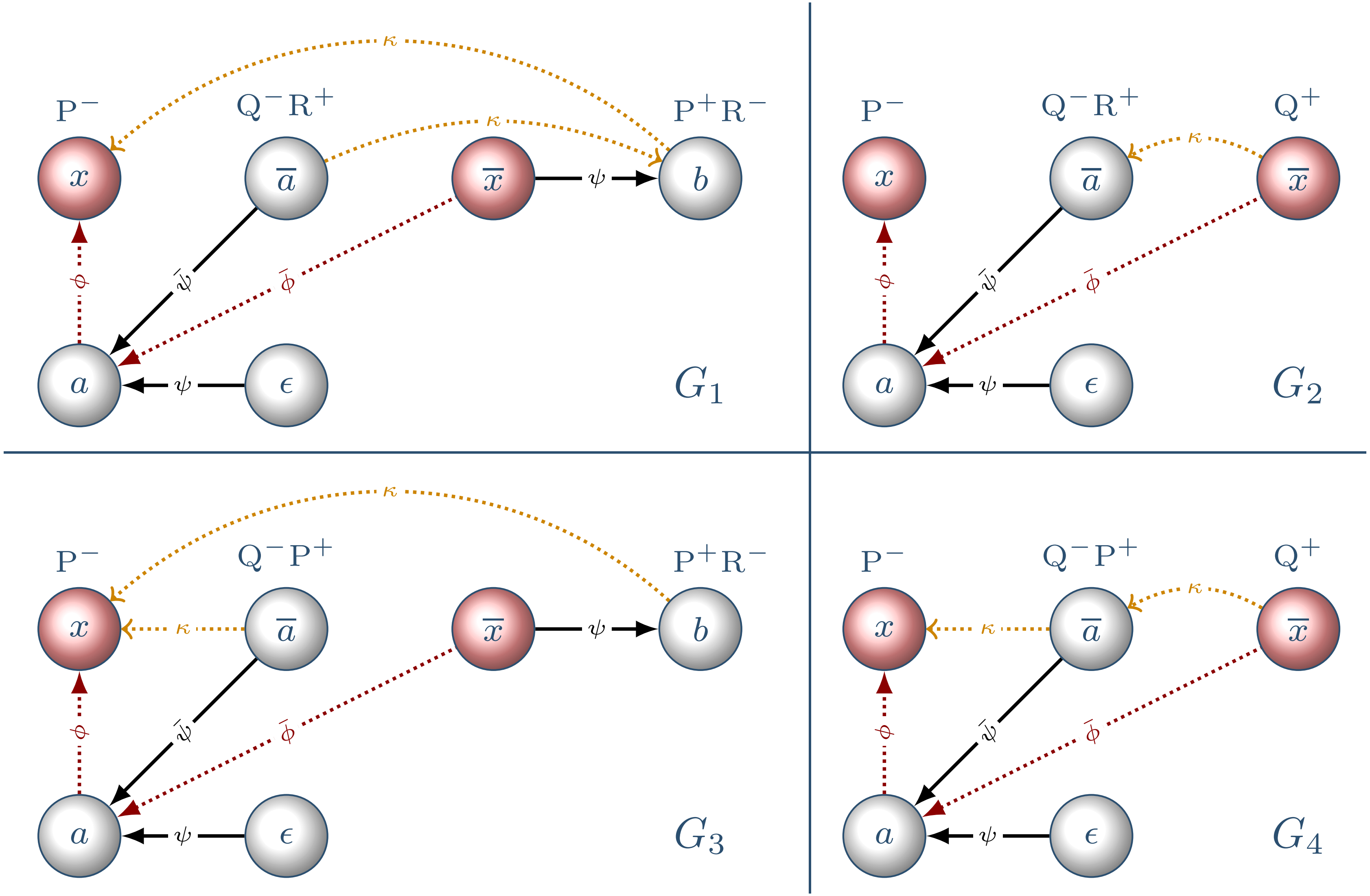

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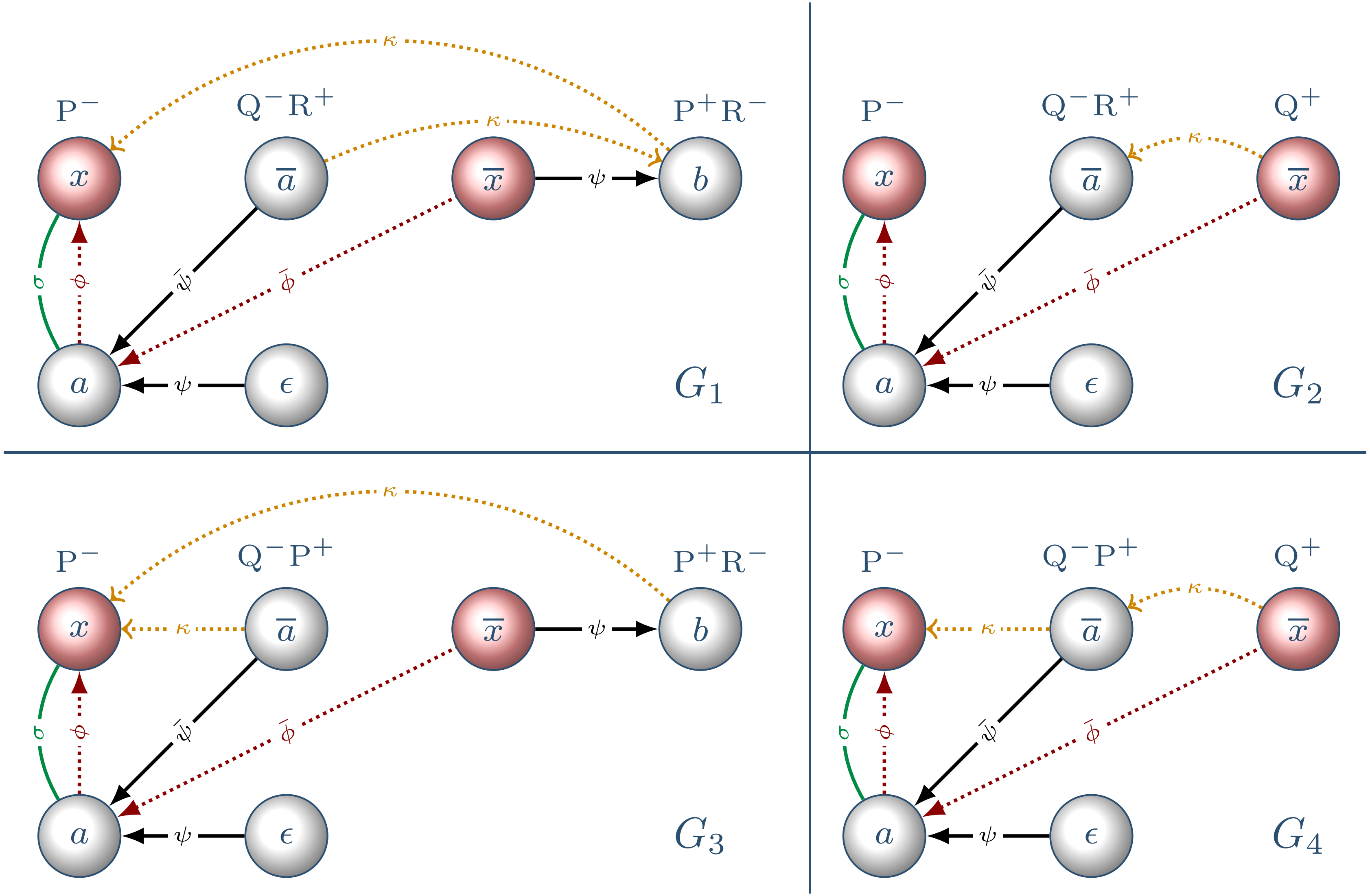

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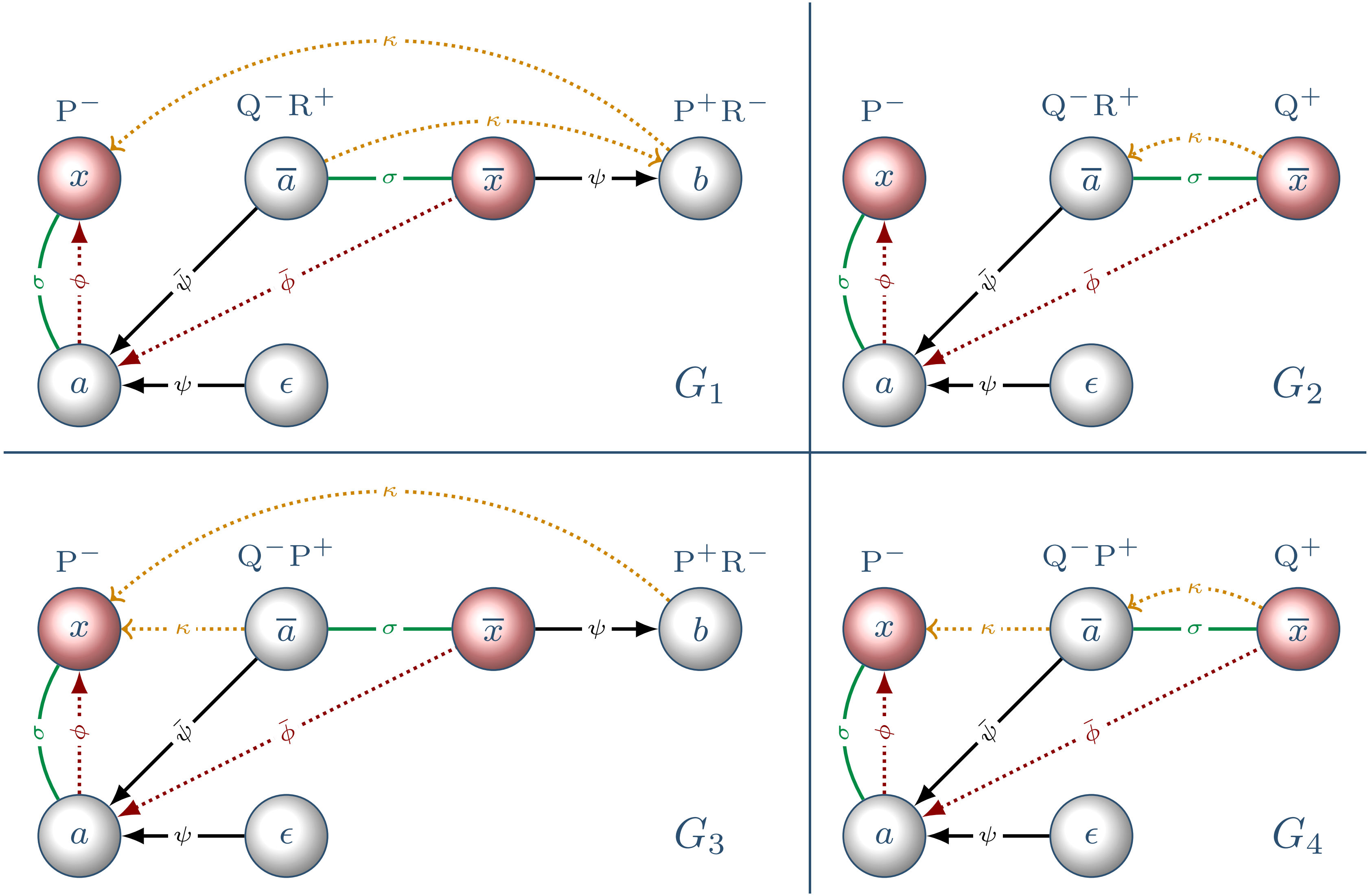

R-Graph Slicing

- S-**slice** of $G(V, E)$: **smallest** biRG $G^S(V^S, E^S)$ such that
  - $S \subseteq V^S$ and for all vertices $u \in V^S$ and $v \in V$
  - if $v[\tau]u \in E$ and $\tau \in \{\psi, \phi, \sigma\}$, then $v \in V^S$ and $v[\tau]u \in E^S$
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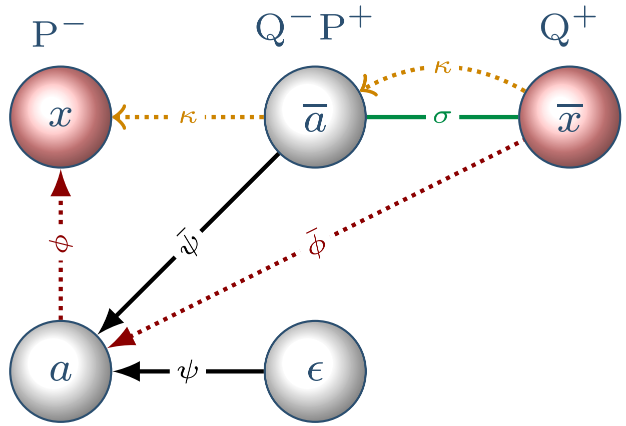
Initial Graph	$\{\overline{a}, \overline{x}\}$-slice
	

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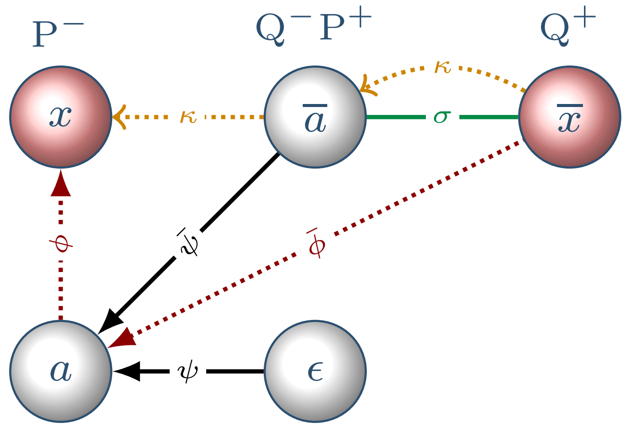	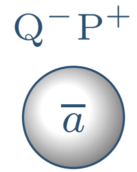

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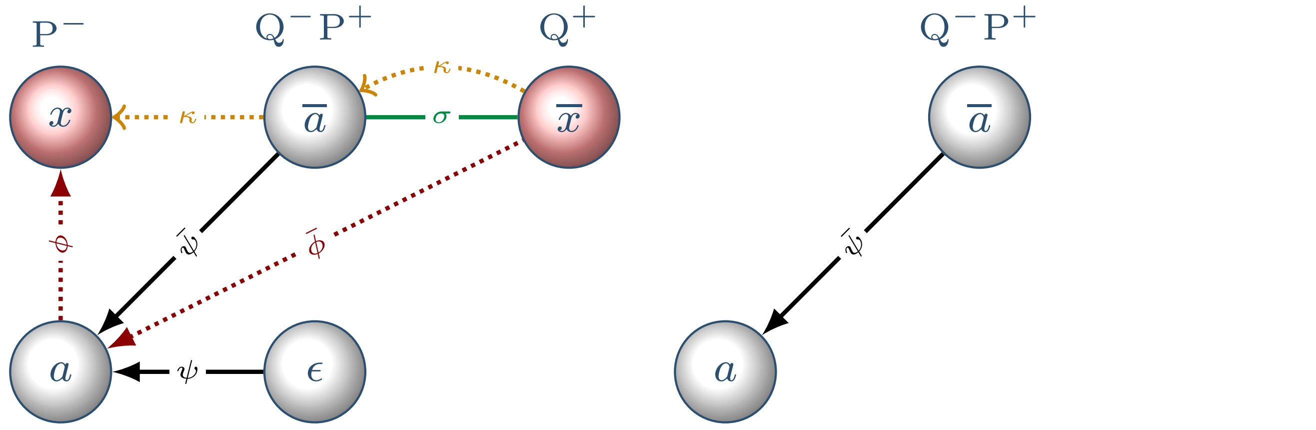

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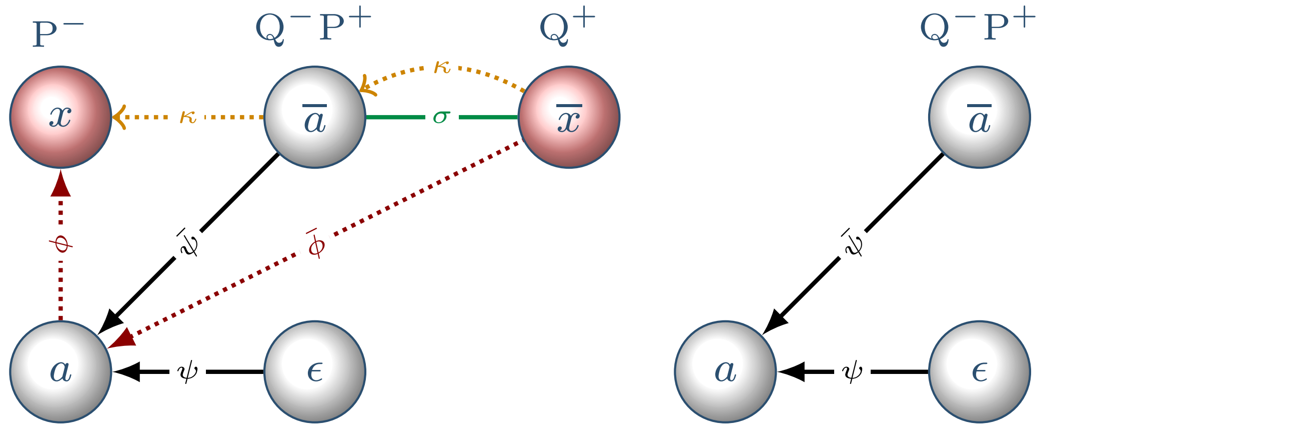

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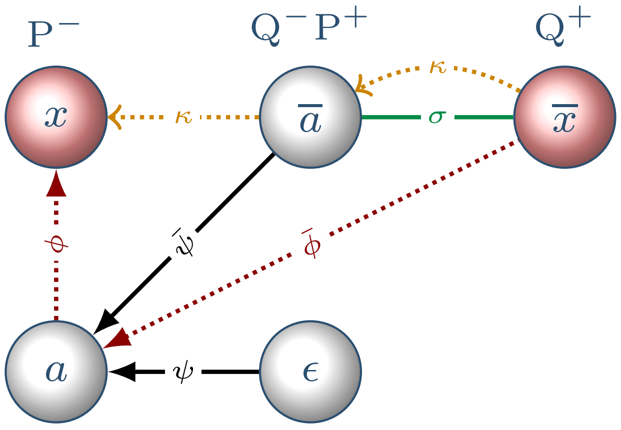	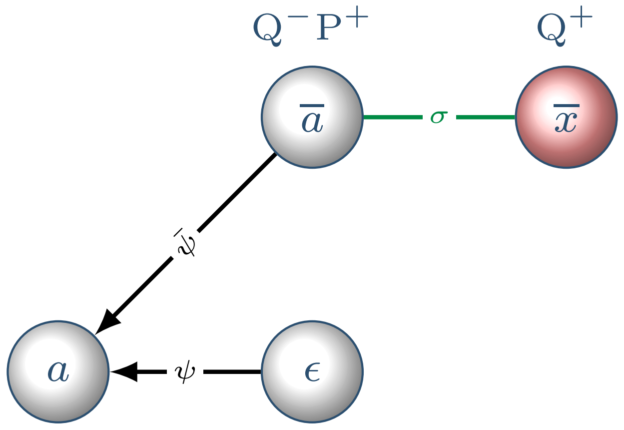

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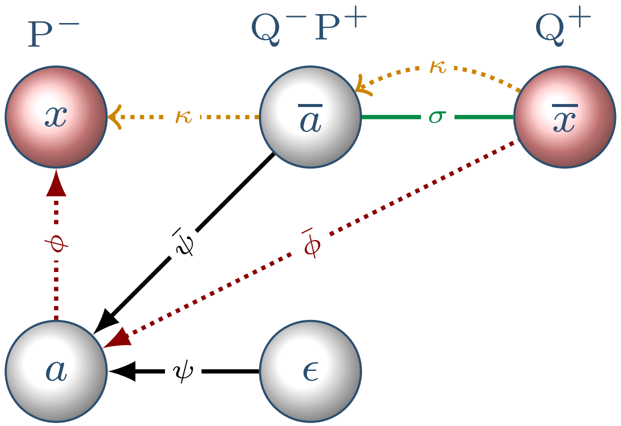	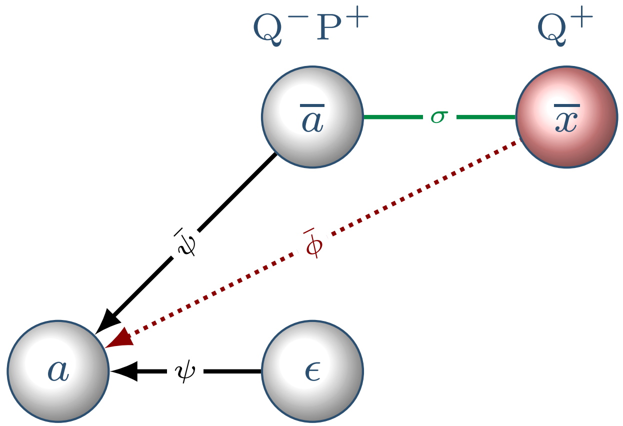

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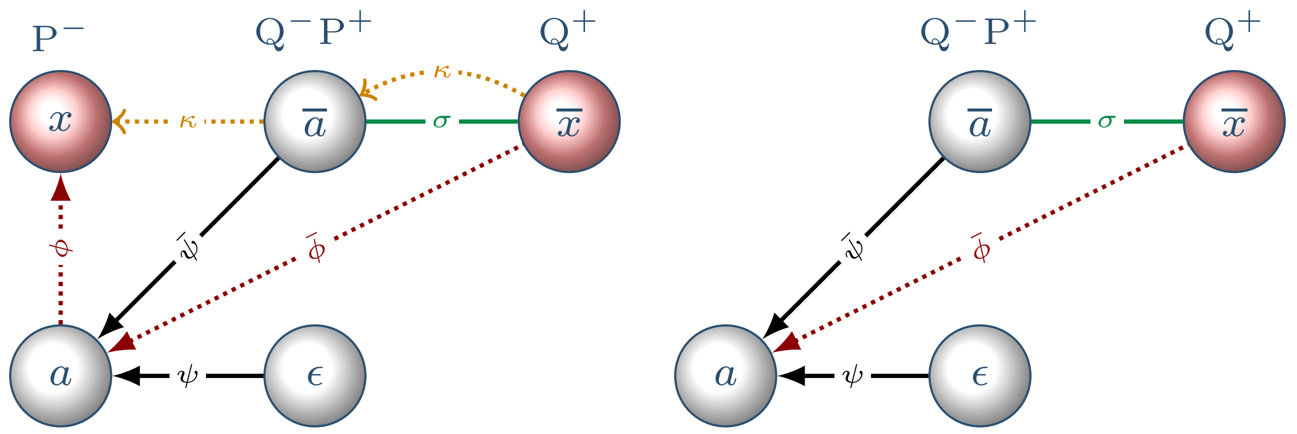

- Only keep **relevant information**

Concrete Paths

Concrete Paths

- T-**path** in $G(V,E)$: sequence $P = u_0\tau_1 u_1 \ldots u_{p-1}\tau_p u_p$ such that
  - $u_0 = u$, $u_p = v$ (T-cycle if $u_0 = u_p$)
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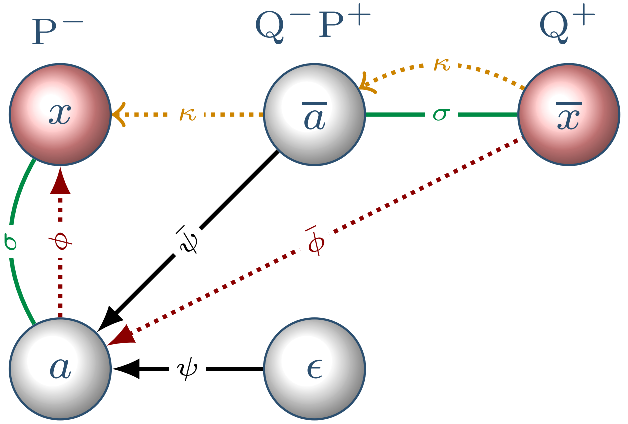
- **Concrete path** iff $T \subseteq \{\psi, \bar{\psi}, \sigma\}$ (only **solid edges**)
- **Virtual path** iff $T \subseteq \{\phi, \bar{\phi}\}$ (only **densely dotted edges**)



$\overline{x}\,\sigma\,\overline{a}\,\bar{\psi}\,a\,\sigma\,x$ is a concrete path

R-Graph Properties

Let $G(V, E)$ be a biRG

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- **Complementarity:** at least one κ-edge $u[\kappa]v$ in E such that

  the slice $G^{\{u,v\}}$ is admissible and $G^{\{u,v\}}u \blacktriangleright v$

biRG-Validity

- A BiInt formula A is biRG-valid iff for some **multiplicity** μ and for the final biRG-collection R_f of **irreducible** biRGs through $\mu(\mathrm{A})$ there exist a (global) σ-binding $\overline{\sigma}$ and a (global) κ-binding $\overline{\kappa}$ such that:

  1. For all (not necessarily distinct) biRGs $G_1(V_1, E_1), G_2(V_2, E_2)$ in $\overline{\sigma} \circ \overline{\kappa}(R_f)$ and all ϕ-vertices x in $V_1 \cap V_2$,

     $$\text{if } x[\sigma]u \in E_1 \text{ and } x[\sigma]v \in E_2 \text{ then } u = v$$

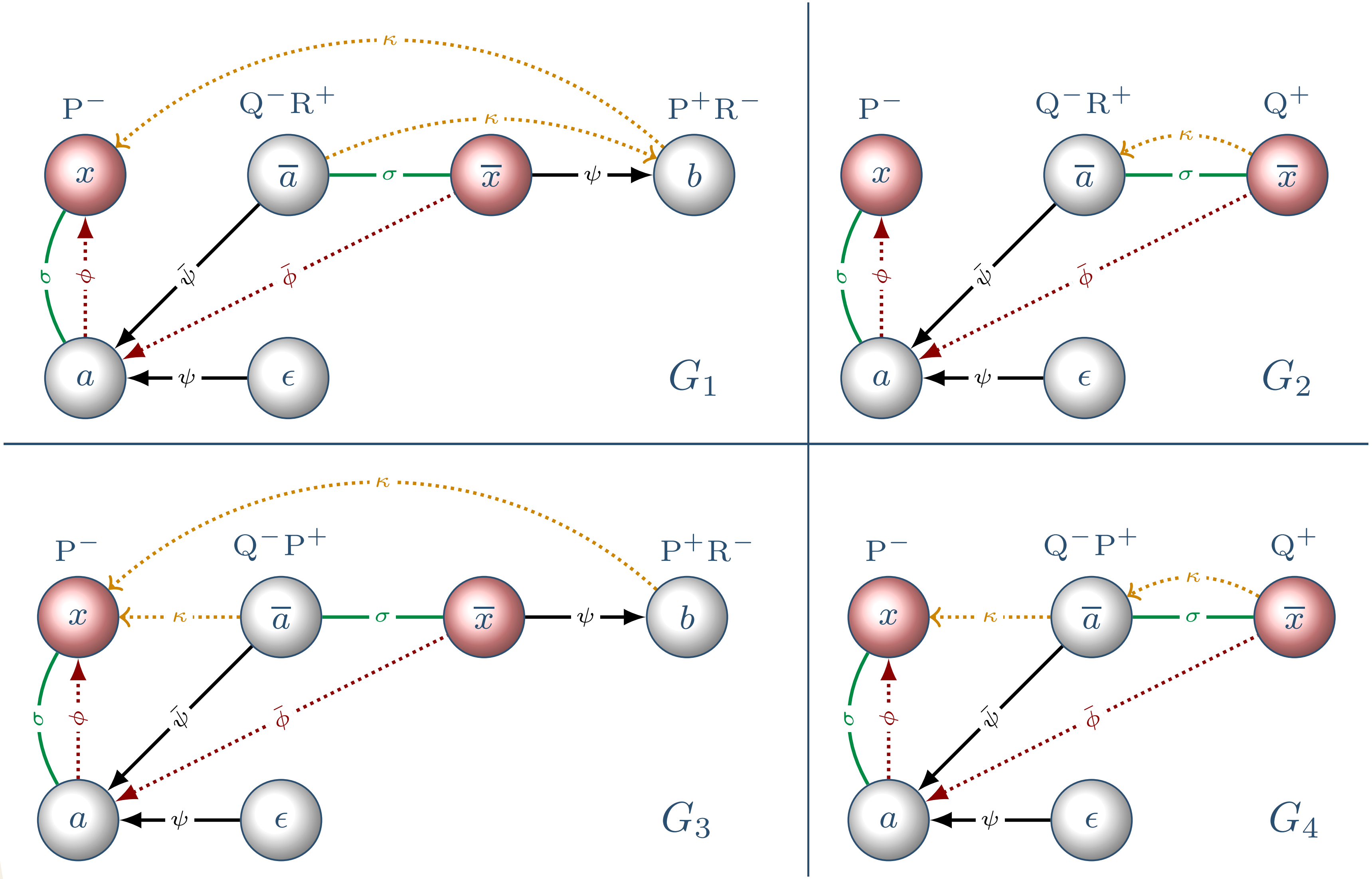
  2. All **consistent** biRGs $G(V, E)$ in $\overline{\sigma} \circ \overline{\kappa}(R_f)$ are **complementary**
  3. The **reduction ordering** $\lhd = (\ll \cup \sqsubset)^+$ induced by $\overline{\sigma}$, where $(\cdot)^+$ stands for transitive closure, is **irreflexive**

- Theorem (soundness and completeness)

  A BiInt formula A is biRG-valid iff it is valid in the Kripke semantics

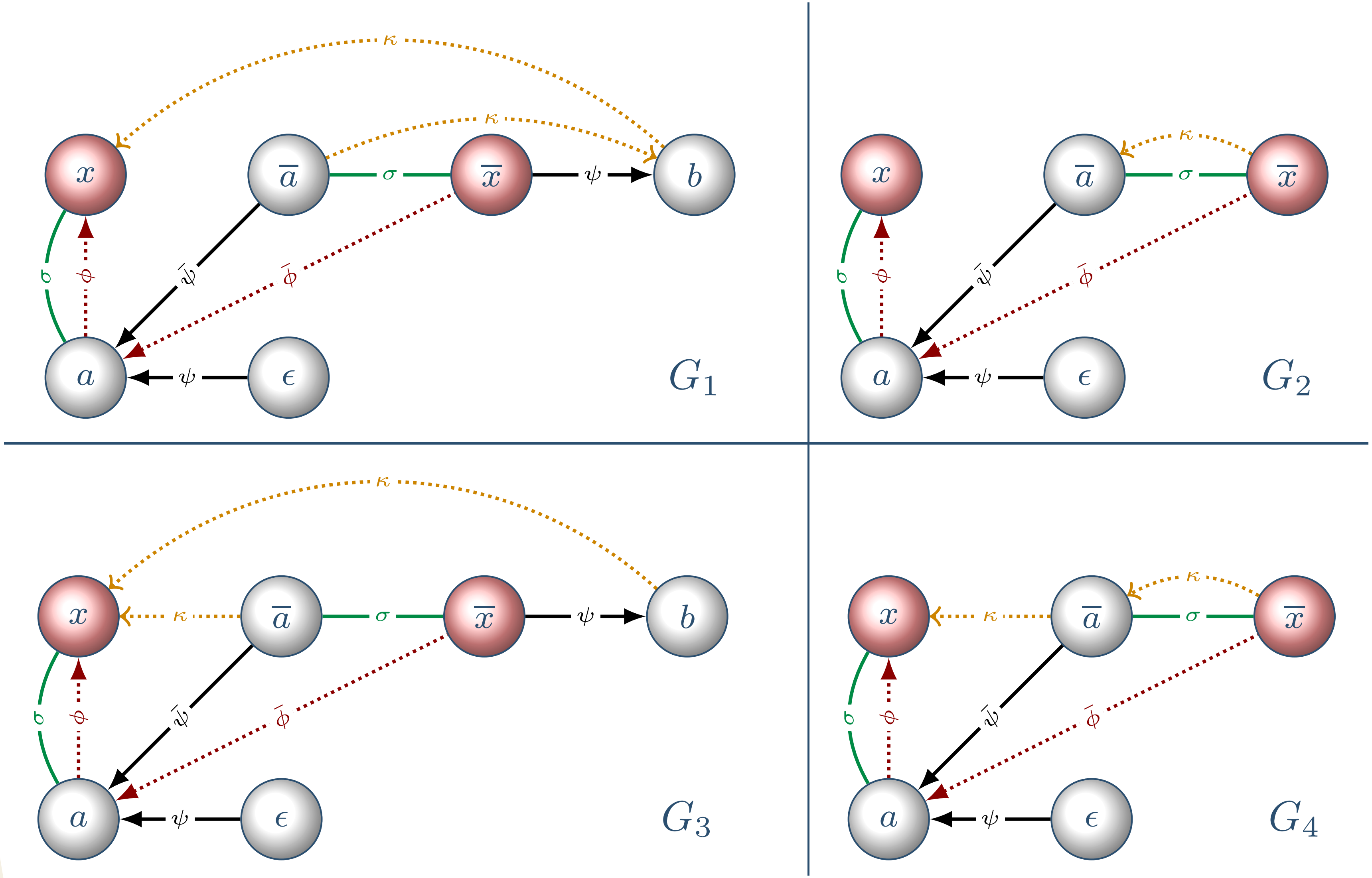
biRG-Validity Example

$$(((\mathrm{R}^+_{2} \vee^+_{1} \mathrm{P}^+_{3}) \prec^+_{\overline{a}} \mathrm{Q}^-_{4}) \wedge^+_{0} \neg^-_{x} \mathrm{P}^-_{5}) \rightarrow^-_{a} ((\mathrm{P}^+_{6} \rightarrow^-_{b} \mathrm{R}^-_{7}) \prec^-_{\overline{x}} \mathrm{Q}^+_{8})$$



biRG-Validity Example

$$(((\mathrm{R}_{2}^{+} \vee_{1}^{+} \mathrm{P}_{3}^{+}) \prec_{\overline{a}}^{+} \mathrm{Q}_{4}^{-}) \wedge_{0}^{+} \neg_{x}^{-} \mathrm{P}_{5}^{-}) \rightarrow_{a}^{-} ((\mathrm{P}_{6}^{+} \rightarrow_{b}^{-} \mathrm{R}_{7}^{-}) \prec_{\overline{x}}^{-} \mathrm{Q}_{8}^{+})$$



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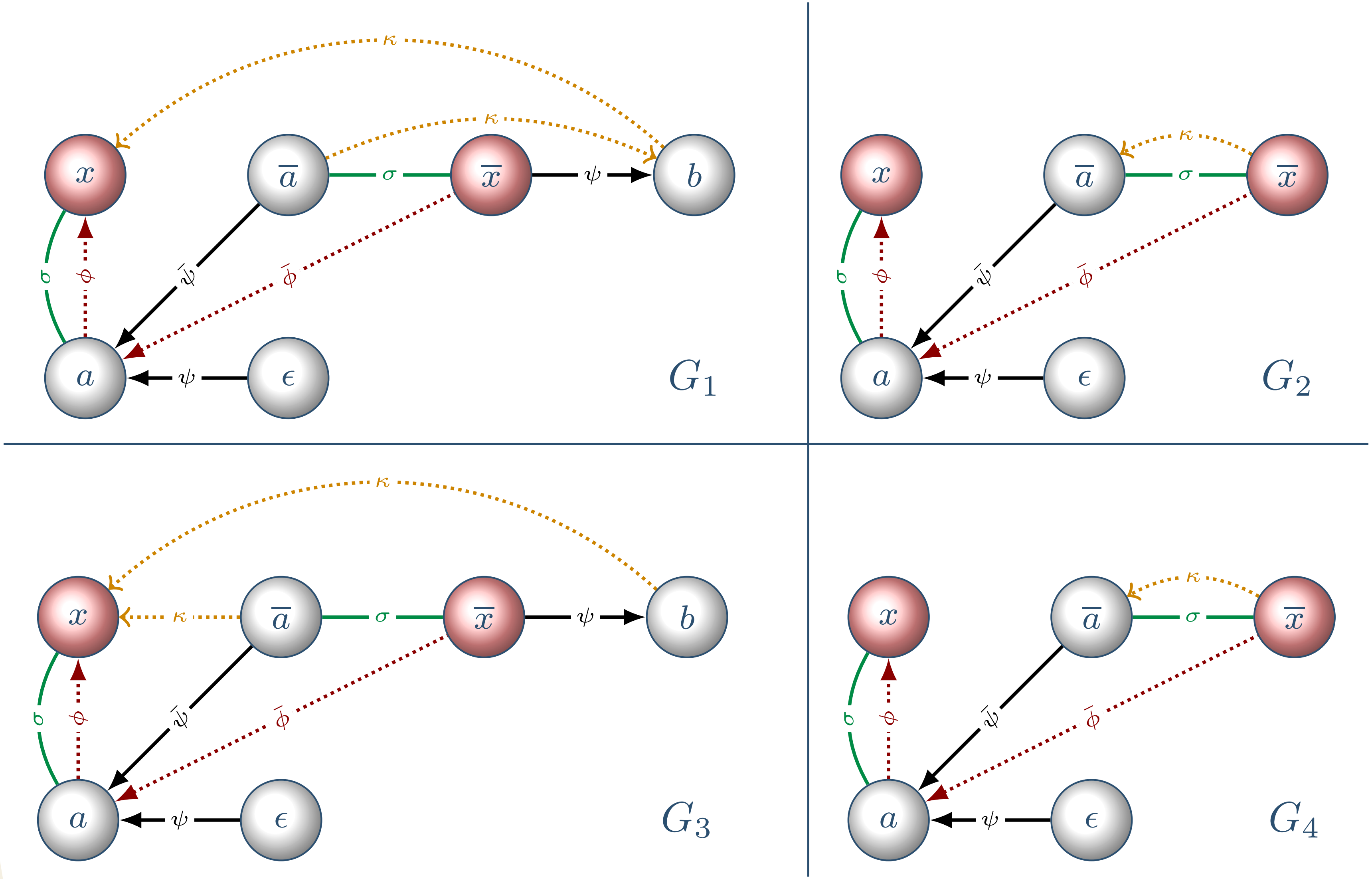

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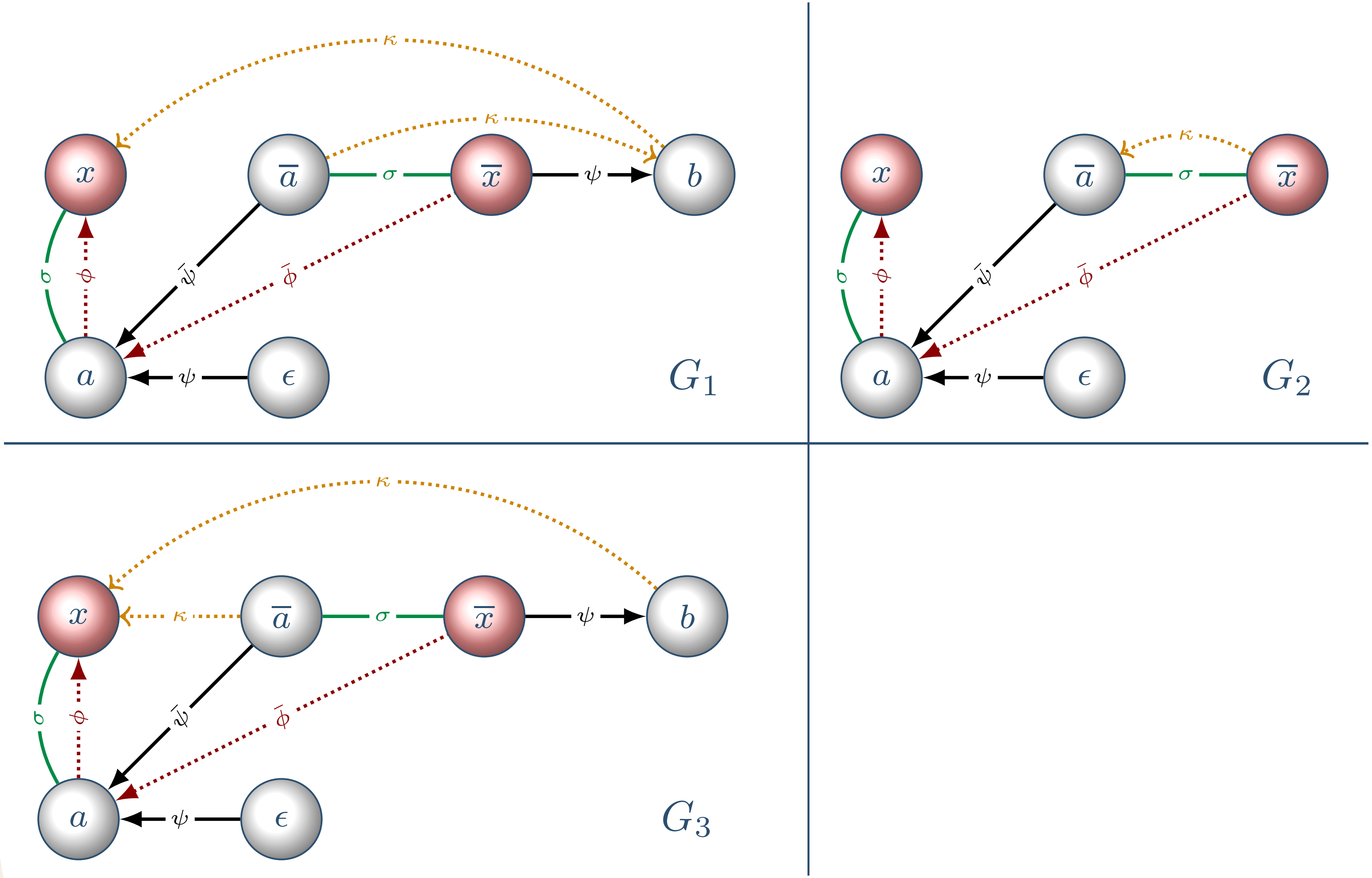

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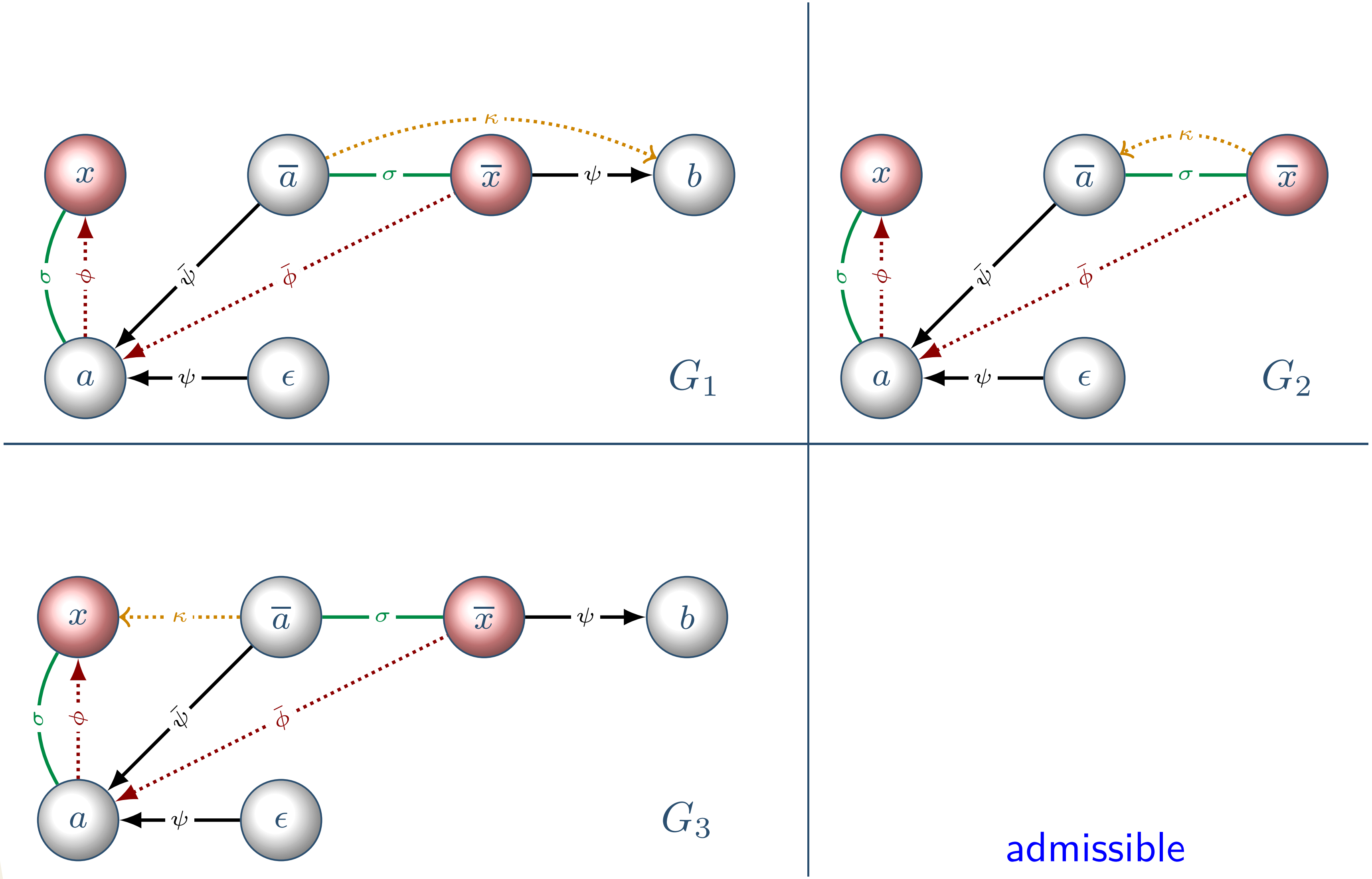

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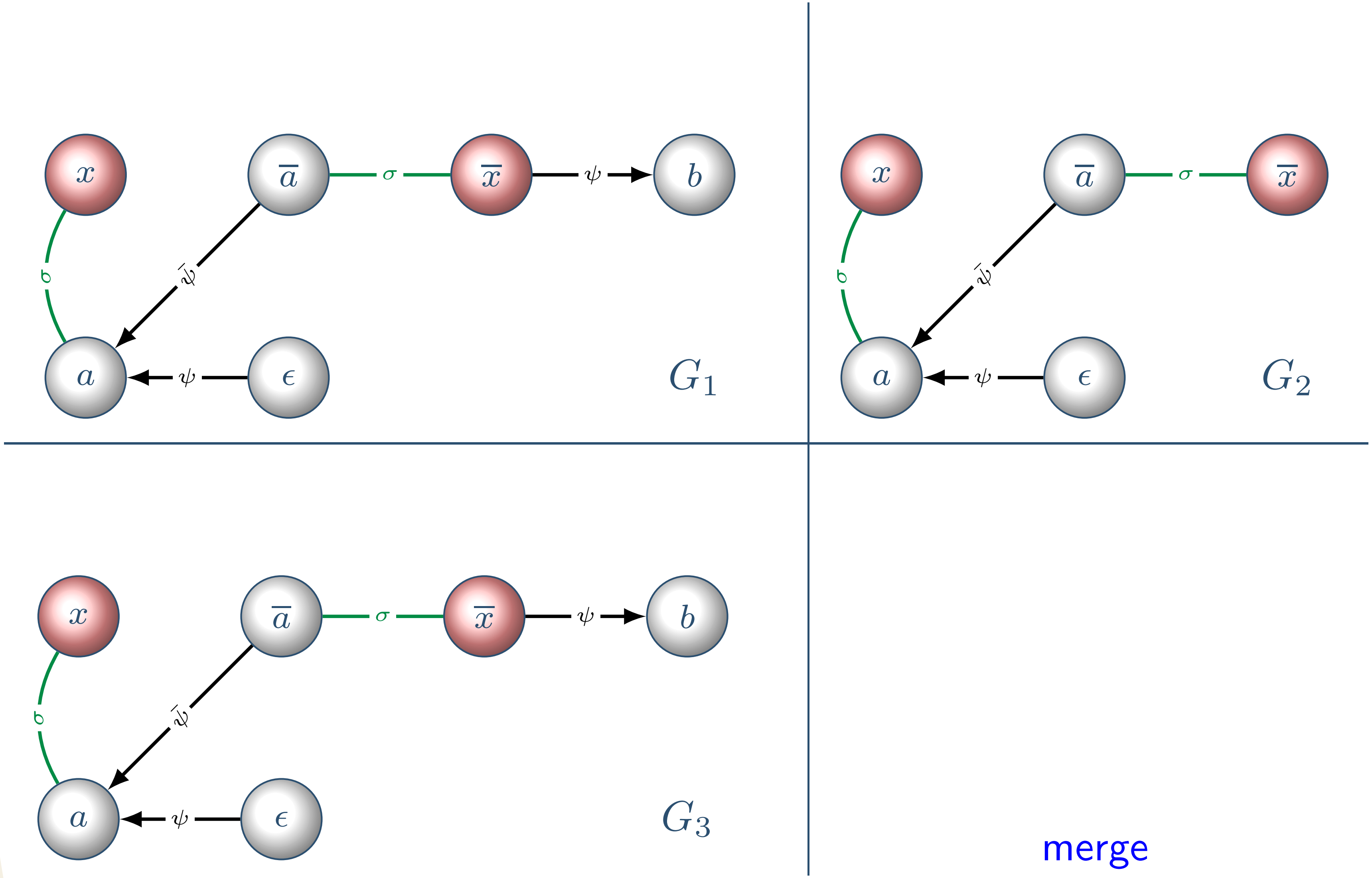

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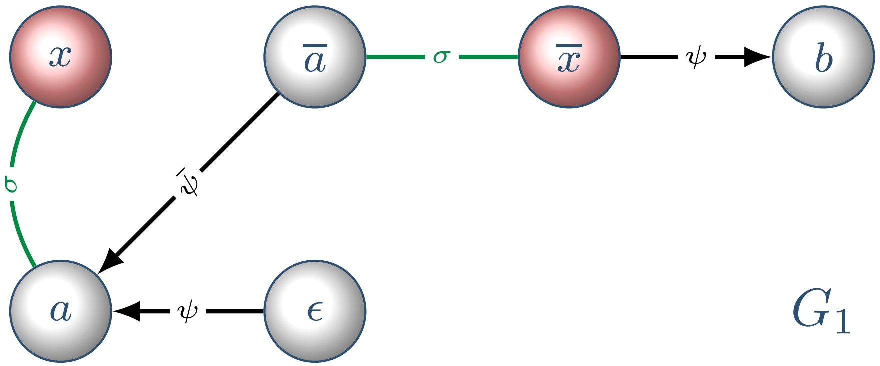

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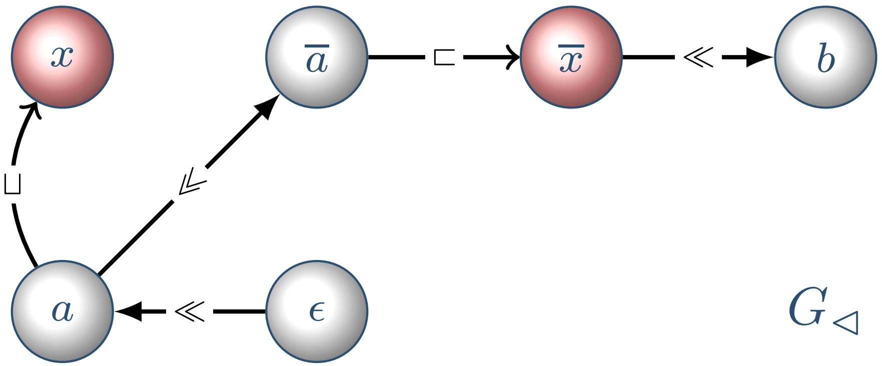


Valid !

Counter-Model Extraction

biRG-Saturation

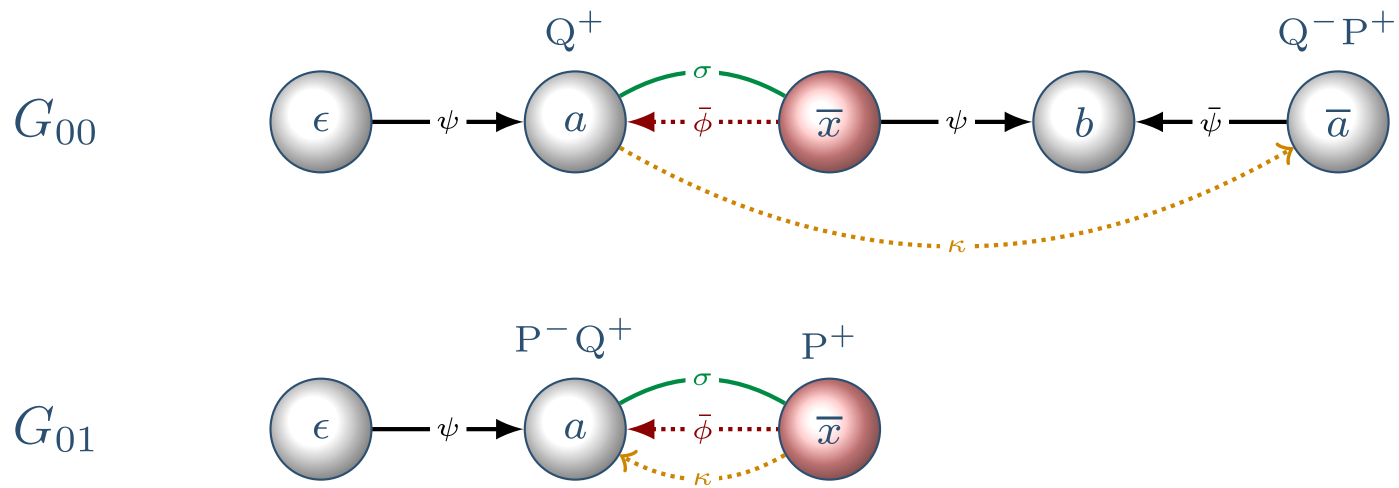
Smallest relation between vertices and signed formulas such that:

- Base case: for all A in $\{\bot\} \cup \mathcal{V}$
  - $Gu \Vdash \mathrm{A}^+$ iff $\exists v(Gv \blacktriangleright u$ and $\mathrm{A}^+ \in \mathcal{F}(G, v))$
  - $Gu \Vdash \mathrm{A}^-$ iff $\exists v(Gu \blacktriangleright v$ and $\mathrm{A}^- \in \mathcal{F}(G, v))$
- Induction:
  - $Gu \Vdash (\mathrm{A} \wedge \mathrm{B})^+$ iff $Gu \Vdash \mathrm{A}^+$ and $Gu \Vdash \mathrm{B}^+$
  - $Gu \Vdash (\mathrm{A} \wedge \mathrm{B})^-$ iff $Gu \Vdash \mathrm{A}^-$ or $Gu \Vdash \mathrm{B}^-$
  - $Gu \Vdash (\mathrm{A} \vee \mathrm{B})^+$ iff $Gu \Vdash \mathrm{A}^+$ or $Gu \Vdash \mathrm{B}^+$
  - $Gu \Vdash (\mathrm{A} \vee \mathrm{B})^-$ iff $Gu \Vdash \mathrm{A}^-$ and $Gu \Vdash \mathrm{B}^-$
  - $Gu \Vdash (\mathrm{A} \to \mathrm{B})^+$ iff $\forall v(Gu \blacktriangleright v$ and $Gv \Vdash \mathrm{A}^+$ imply $Gv \Vdash \mathrm{B}^+)$
  - $Gu \Vdash (\mathrm{A} \to \mathrm{B})^-$ iff $\exists v(Gu \blacktriangleright v$ and $Gv \Vdash \mathrm{A}^+$ and $Gv \Vdash \mathrm{B}^-)$
  - $Gu \Vdash (\mathrm{A} \prec \mathrm{B})^+$ iff $\exists v(Gv \blacktriangleright u$ and $Gv \Vdash \mathrm{A}^+$ and $Gv \Vdash \mathrm{B}^-)$
  - $Gu \Vdash (\mathrm{A} \prec \mathrm{B})^-$ iff $\forall v(Gv \blacktriangleright u$ and $Gv \Vdash \mathrm{A}^+$ imply $Gv \Vdash \mathrm{B}^+)$

Counter-Models (1)

Let's start with $\mathrm{D} = \mathrm{Q} \rightarrow ((\neg(\mathrm{P} \prec \mathrm{Q}) \prec \mathrm{P}) \vee \mathrm{P})$

- Two **irreducible** biRGs through D with multiplicity 1

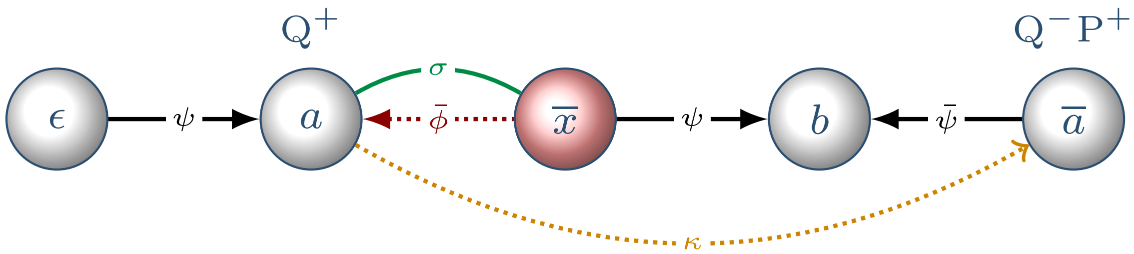


- Both **admissible** and **consistent**, but G_{00} is **not complementary**
- if G_{00} is also **saturated**, extract **counter-model**
- G_{00} is saturated !

Counter-Models (2)

Let $\mathrm{D} = \mathrm{Q} \rightarrow ((\neg(\mathrm{P} \prec \mathrm{Q}) \prec \mathrm{P}) \vee \mathrm{P})$

G_{00}



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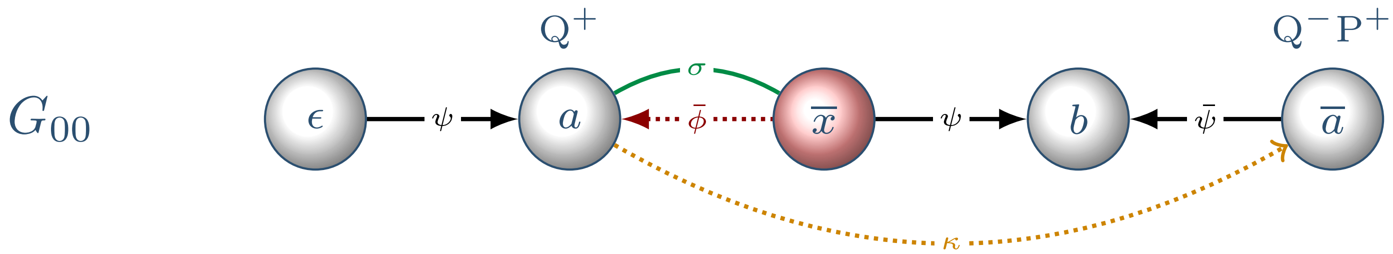

- Quotient: σ-edges $\rightsquigarrow$ **vertex-classes**

$$V = \{\dot{\epsilon}, \dot{a}, \dot{b}, \dot{\overline{a}} \mid \dot{\epsilon} = \{\epsilon\}, \dot{a} = \dot{\overline{x}} = \{a, \overline{x}\}, \dot{b} = \{b\}, \dot{\overline{a}} = \{\overline{a}\}\}$$

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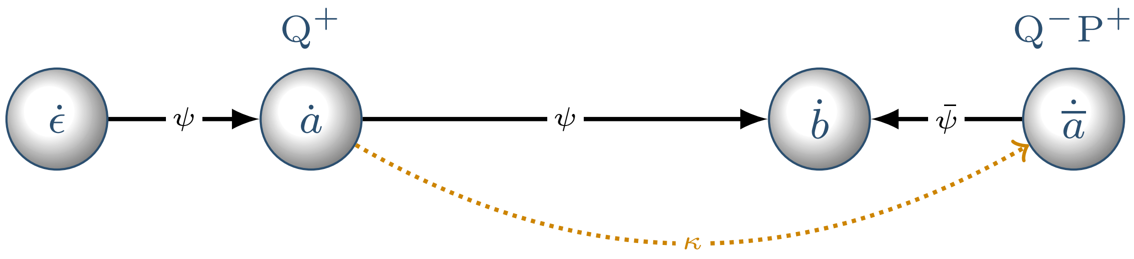

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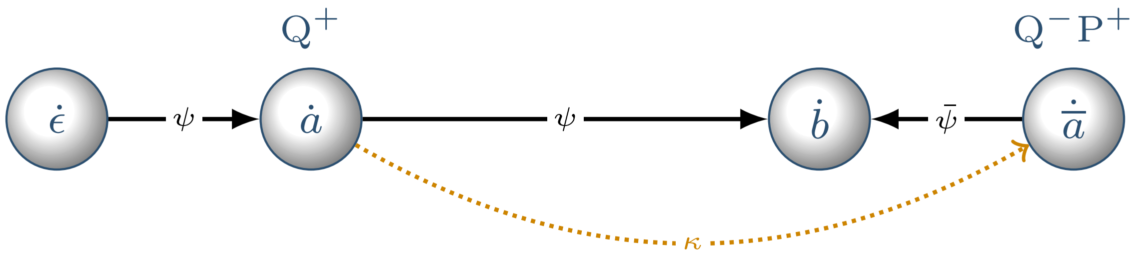

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$$(\forall m, n \in V)(m \sqsubseteq n \text{ iff } (\exists m' \in m)(\exists n' \in n)(Gm' \blacktriangleright n'))$$

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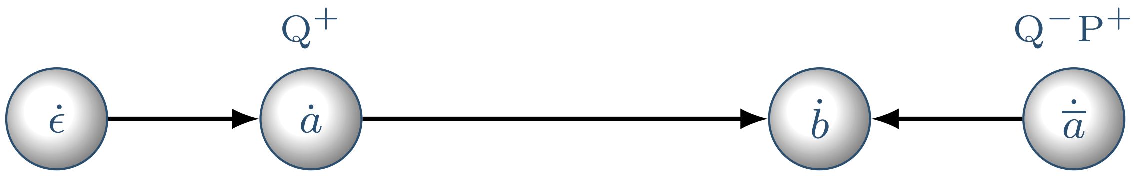

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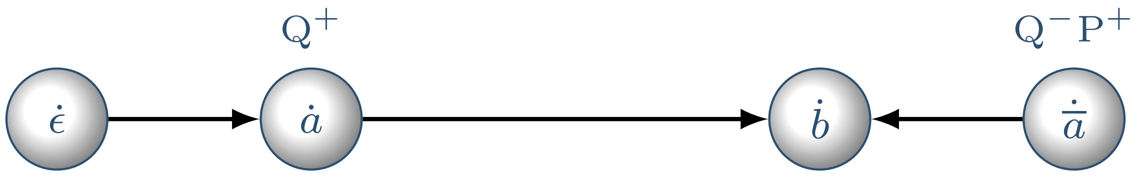

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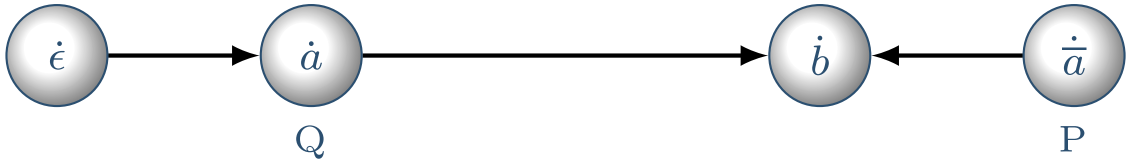

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Binding & Splitting Variables

Solving Admissibility Constraints (1)

How to find admissible σ-bindings ?

- Root $R(u) =$ first vertex introducing variable u in a reduction
- Domain $D(u) =$ all suitable instantiations (σ-links) for u

Initial step

- Compute $R(u)$ and $D(u)$ for all variables in $G(V, E)$
- Sort variables in a partially ordered list $X = x_1, \ldots, x_n$ such that

$$\text{if } i < j \text{ and } x_i \propto x_j \text{ then } x_j \propto x_i$$

where $x_i \propto x_j$ (x_i depends on x_j) iff $x_i \in D(x_j)$

Solving Admissibility Constraints (2)

Enumeration step

- for each variable $u = x_i$ in X, select a constant c in $D(u)$ and
- perform
  - applyRule := selectRule(u,c,v)
  - $R(v), D(v)$:= applyRule(u,c,v)

  on all variables $v = x_j$ in X such that $j > i$
- if all variables in X can be bound: **admissible σ-binding**
- otherwise: backtrack and change selections

Solving Admissibility Constraints (3)

- Selecting and applying a rewrite rule

Rule	selectRule(u,c,v)
Bind	$R(v) = u$
Narrow	$c \in A(v)$
Widen	$c = R(v) \wedge c \neq R(u) \wedge itype(u) = itype(v)$

Rule	applyRule(u,c,v)
Bind	$c, D(v) \cup M_t(c, v)$
Narrow	$R(v), D(v) - A(u)$
Widen	$R(v), D(v) \cup M_t(u, v)$

- $A(u)$ = all constants introduced after u
- $S(u)$ / $P(u)$ = all constants that are successors / predecessors of u
- $M_t(u, v) = M_t(u) - A(v)$ with $M_\phi = S$ and $M_{\overline{\phi}} = P$

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- Variable **splitting** $=$ variable **sharing + splitting ordering**
  - splitting set S: downward closed dual-free β_1 or β_2 indexes
  - colored variables x^S and substitutions σ^S
  - splitting ordering $\prec$

    $$\sigma(x^X) \neq \sigma(x^Y) \Rightarrow (\exists i \in X)(\exists j \in Y)((i \triangle j) \prec x)$$

  - validity: $\lhd = (\ll \cup \sqsubset \cup \prec)^+$ is irreflexive

Conclusion

Concluding Remarks

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- Our results
  - ❖ New characterization of BiInt validity through R-Graphs
  - ❖ New labelled sequent calculus with FV (variable splitting)
  - ❖ Counter-model construction + cut-elimination (completeness)
  - ❖ New algorithm to solve $\sigma\kappa$-bindings (generalization of IL prefix-unification)

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- Our results
  - New characterization of BiInt validity through R-Graphs
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  - Counter-model construction + cut-elimination (completeness)
  - New algorithm to solve $\sigma\kappa$-bindings (generalization of IL prefix-unification)
- Future and on-going work
  - Implementation of an ATP for BiInt
  - $O(n\,log\,n)$ space complexity (graph superposition)
  - Similar calculi for richer resource logics